30 A Short Introduction to Power in AC Circuits

v1.5 – June 2021

Prerequisite knowledge required: Phasors, AC Circuit Analysis, DC Circuit Analysis

30.1 Introduction

In DC circuits the formula for power is simple:

$$P = V \times I \tag{30.1}$$

The power dissipated or supplied by a component is the product of the voltage across the component multiplied by the current flowing through it.

If the current is flowing from a lower voltage to a higher voltage, then since it takes work to push the charge against the force of the electric field, the component must be suppling power to the circuit. On the other hand if the current is flowing from the higher voltage to the lower voltage, then energy is being released from the circuit, and this energy must be being dissipated somehow (for example being converted to heat in a resistor, or light in an LED).

If the component is a resistor and obeys Ohm's law, it's straightforward to derive that the power dissipated in the resistor (of resistance *R*) is:

$$P = V \times I = \frac{V^2}{R} = I^2 R \tag{30.2}$$

As far as DC circuits go, that's about all there is to know.

In AC circuits things are slightly more complicated, since the voltage and current are usually represented by phasors, and these phasors might not lie in the same direction (in other words, the voltage across the component and the current flowing through it might not be in phase). This introduces a few complications. However, since most power distribution is AC[1], it is important to understand the issues and their consequences.

This chapter explores the issue of power in AC circuits, in particular looking at what happens in reactive components such as capacitors and inductors, introduces the concepts of active, reactive and apparent power, and discusses the maximum power theorem as applied to AC circuits.

30.2 AC power in resistors

In phasor terms, the voltage across a resistor is usually specified in terms of the zero-to-peak amplitude[2] of the voltage across the resistor (so that the magnitude of a phasor is the maximum voltage or current of the real signal represented by that phasor). However you can't determine the average AC power in a resistor by multiplying the zero-to-peak amplitude of the voltage by the zero-

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[1] At least it is at the moment, for example the domestic power supply in the UK is 240 V rms at 50 Hz, although there is an increasing amount of interest in high-voltage DC systems.

[2] At least it is in this chapter. There is such a thing as an rms phasor (which has an amplitude equal to the rms amplitude of the waveform), but all phasors in this chapter are zero-to-peak.

to-peak amplitude of the current. To see why, consider a plot of the voltage across the resistor against the current flowing through it:

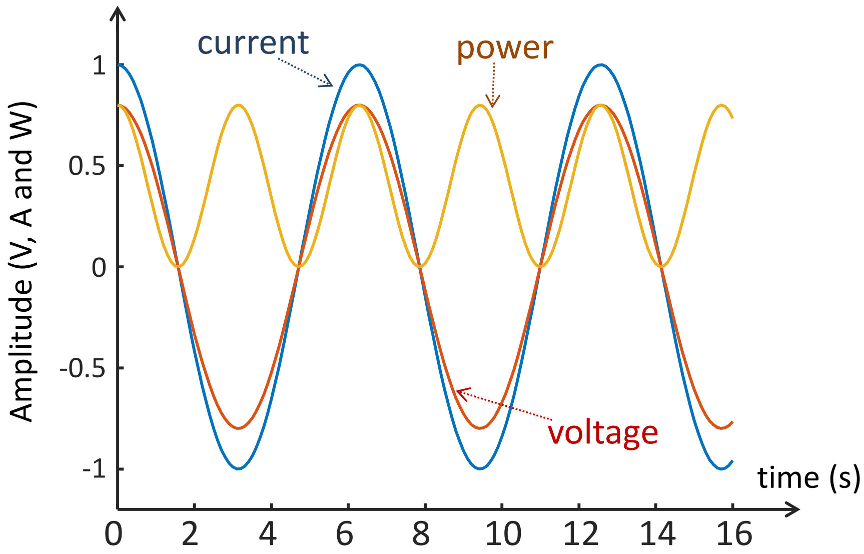

Figure 30.1 Voltage, current and instantaneous power in a resistor

If the maximum value of the current (the magnitude of the phasor) is I_0, the current through the resistor could be written:

$$I(t) = I_0 \cos(\omega t) \tag{30.3}$$

and similarly since the voltage and current are in-phase for a resistor, the voltage across the resistor could then be written:

$$V(t) = V_0 \cos(\omega t) \tag{30.4}$$

where V_0 is the magnitude of the phasor representing the voltage. The instantaneous power (the power dissipated at every instant of time) is then given as always by the product of the voltage and the current:

$$\begin{aligned} P(t) &= V_0 \cos(\omega t) \times I_0 \cos(\omega t) \\ &= V_0 I_0 \cos^2(\omega t) \\ &= \frac{V_0 I_0}{2} + \frac{V_0 I_0}{2}\cos(2\omega t) \end{aligned} \tag{30.5}$$

Since the average value of the cosine is zero, the average power in terms of the phasor amplitudes is therefore:

$$\overline{P(t)} = \frac{V_0 I_0}{2} = \frac{V_0^{\,2}}{2R} = \frac{I_0^{\,2} R}{2} \tag{30.6}$$

which in terms of the phasors themselves could be written:

$$\overline{P(t)} = \frac{|\mathbf{V} \times \mathbf{I}|}{2} = \frac{|\mathbf{V}|^2}{2R} = \frac{|\mathbf{I}|^2 R}{2} \tag{30.7}$$

The additional factor of two is required in AC power circuit analysis to allow for the fact that the voltage and the current do not have their maximum values all the time. They oscillate up and down, and most of the time the voltage and current have less than their maximum values.

This average power is known as the *active power* and corresponds to power actually leaving the circuit (in the case of resistors, in the form of heat). The resistor really does heat up; the voltage source providing the voltage really does require to provide that energy from somewhere external to the circuit (e.g. the chemical processes in a battery).

(That might sound obvious, but as we'll soon see there is another type of power that doesn't involve any energy entering or leaving the circuit.)

(Another way to do this calculation is to use the rms voltage and rms current instead of the zero-to-peak voltage and zero-to-peak current. If the rms quantities are being used then the average power dissipated is:

$$\overline{P(t)} = V_{rms} I_{rms} = \frac{V_{rms}{}^2}{R} = I_{rms}{}^2 R \tag{30.8}$$

since:

$$V_{rms} = \frac{V_0}{\sqrt{2}} \qquad I_{rms} = \frac{I_0}{\sqrt{2}} \tag{30.9}$$

There is something called an rms phasor, which has an amplitude equal to the rms amplitude of the voltage or current, but the same phase angle as the normal zero-to-peak phasor, and using these rms phasors removes the need for the factor of ½ in equation (30.6), however in these notes I'll stick to using phasors with amplitudes equal to the zero-to-peak amplitudes of the voltages or currents.)

30.3 AC power in capacitors

With capacitors it's easiest to start thinking about the actual real currents and voltages, rather than their phasor representations. If the real current through[3] a capacitor is:

$$I(t) = I_0 \cos(\omega t) \tag{30.10}$$

(which could be represented by a phasor with a phase of zero and an amplitude of I_0), the voltage across the capacitor's terminals $V_C(t)$ can be determined from:

[3] Although please remember that current doesn't really flow *through* a capacitor; what's happening is that some charge arrives on one plate, and an equal amount charge leaves the other plate. But it's not the same charge: charge doesn't flow through an ideal capacitor. However I'll use the term *through* for consistency with resistors and inductors.

$$I(t) = \frac{dQ(t)}{dt} = C\frac{dV_C(t)}{dt} \tag{30.11}$$

$$\begin{aligned} \frac{dV_C(t)}{dt} &= \frac{I(t)}{C} = \frac{I_0}{C}\cos(\omega t) \\ V_C(t) &= \frac{I_0}{\omega C}\sin(\omega t) \end{aligned} \tag{30.12}$$

The instantaneous power associated with the capacitor is therefore:

$$P(t) = V_C(t)I(t) = \frac{I_0^2}{\omega C}\sin(\omega t)\cos(\omega t) = \frac{I_0^2}{2\omega C}\sin(2\omega t) \tag{30.13}$$

Plotting the voltage, current and instantaneous power for this capacitor reveals the following:

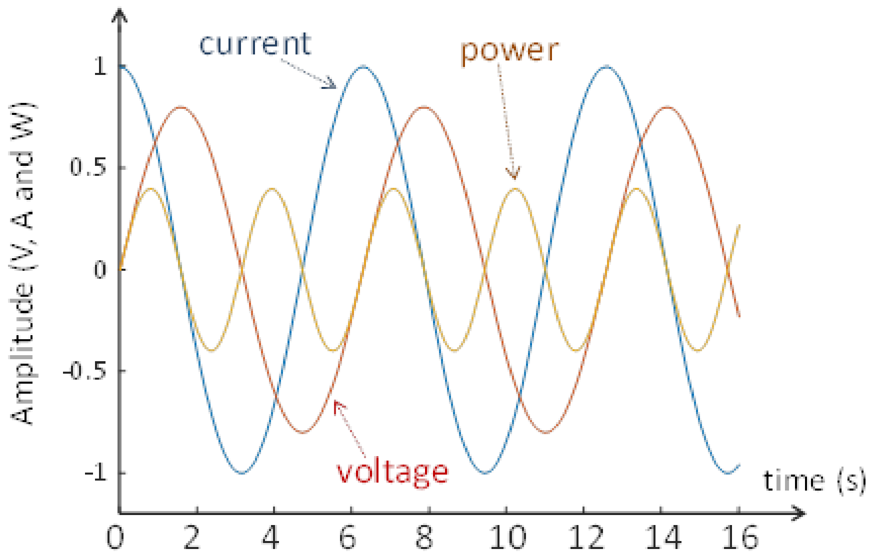

Figure 30.2 Voltage, current and instantaneous power in a capacitor

The important point to note here is that while the *instantaneous power* associated with the capacitor is constantly changing, the average value of the power dissipated by the capacitor is zero, since the average value of the sine function is zero. However, at any given point in time, there will be a non-zero instantaneous power associated with the capacitor that can be either positive or negative: energy is constantly oscillating between flowing onto and then away from the capacitor.

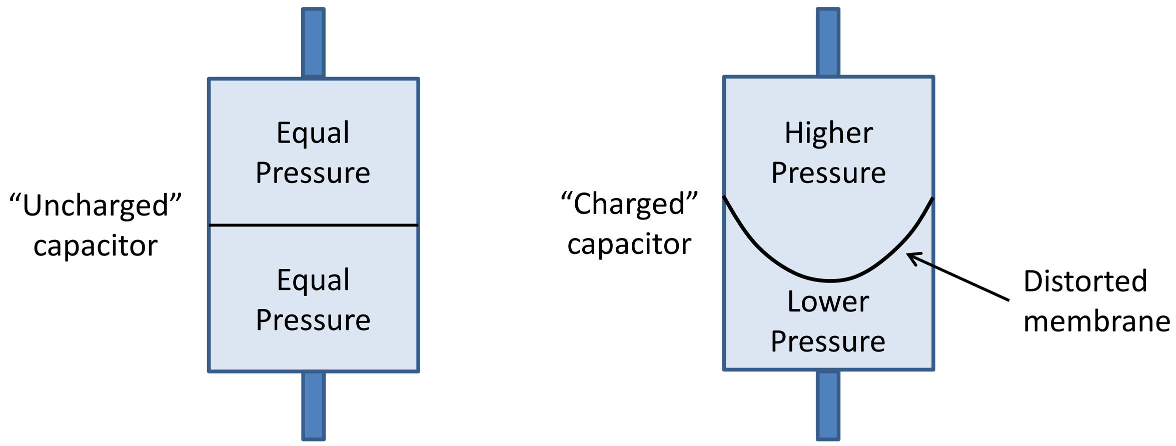

Figure 30.3 "Charged" and "Uncharged" capacitors

Thinking about the capacitor in terms of the hydraulic model (a barrel with an elastic membrane across the middle) this makes some sense: during the times when the capacitor is charging up, it is extracting energy from the circuit, and storing it in the elastic extension of the membrane (in a real capacitor the energy is being stored in the electric field between the plates). Then when the capacitor is being discharged, the instantaneous power being dissipated is negative: the energy stored in the capacitor is being released back into the circuit as the tension in the membrane reduces back to zero (the electric field between the plates reduces) . At these times the capacitor is effectively acting like a battery: a source of energy.

This power is known as *reactive power* since it is associated with reactive components. It doesn't come from outside the circuit, it doesn't get dissipated and lost from the circuit, it just represents an oscillating flow of energy into the electric field and then back out again later in the AC cycle.

Note that in terms of the complex impedance of the capacitor Z_C, this reactive power could be expressed as:

$$P(t) = j\frac{I_0^2 Z_C}{2}\sin(2\omega t) \tag{30.14}$$

30.4 AC power in inductors

For an inductor with a sinusoidal-varying current flowing through it, the voltage across the inductor would be:

$$V_L = L\frac{dI(t)}{dt} = -L\omega I_0 \sin(\omega t) \tag{30.15}$$

which makes the instantaneous power associated with the inductor:

$$P(t) = V_L(t)I(t) = -L\omega I_0 \sin(\omega t) I_0 \cos(\omega t) = -\frac{I_0^2 \omega L}{2}\sin(2\omega t) \tag{30.16}$$

which could be written in terms of the complex impedance of the inductor ($Z_L = j\omega L$) as:

$$P(t) = j\frac{I_0^{\ 2} Z_L}{2}\sin(2\omega t) \tag{30.17}$$

This is similar to the case with capacitors, except that the voltage across the inductor now leads the current by 90 degrees (rather than lagging by 90 degrees), so the plot of the voltage across the inductor to the current flowing through it would look like:

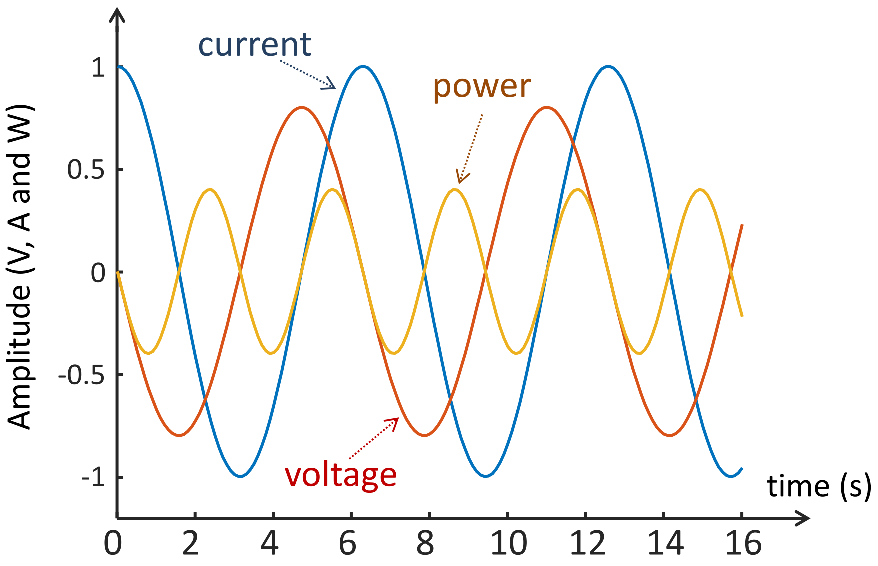

Figure 30.4 Voltage, current and instantaneous power in an inductor

Here the inductor is alternately taking energy out of the circuit to store in its magnetic field, and then returning it back into the circuit. Once again, over time, the average amount of energy being removed from or introduced into the circuit is zero, and this is reflected in the fact that the average value of the instantaneous power is zero.

This is also a form of *reactive power*, this time representing an oscillating flow of energy into the magnetic field of the inductor and back out again.

Notice that in both cases where the impedance is purely reactive (i.e. the complex impedance is purely imaginary), this reactive power is given by:

$$P(t) = j\frac{I_0^{\ 2} Z}{2}\sin(2\omega t) \tag{30.18}$$

where Z is the impedance of the reactive component.

30.5 AC power in a passive network

Having worked out what happens in the case of resistors, capacitors and inductors, the next step is to work out the AC power for any arbitrary network of linear passive components.

The first stage in this calculation is to compute the total impedance of the network at the frequency of interest, for example:

$$Z = R + jX \tag{30.19}$$

where Z is the total impedance, R the real part of the impedance (the resistance) and X the imaginary part of the impedance (the reactance). Any passive linear network can be represented in this way[4].

If X is negative, then a capacitor can be found with the same reactance, by choosing:

$$-jX = \frac{1}{j\omega C} \qquad\qquad C = \frac{1}{\omega X} \tag{30.20}$$

and if X is positive, an inductor can be found with the same reactance:

$$jX = j\omega L \qquad\qquad L = \frac{X}{\omega} \tag{30.21}$$

Placing a resistor of resistance R and either a capacitor or an inductor in series would then give an equivalent network with the same impedance as the given passive network. Since these two components are in series, the same current must be flowing through them (the current that enters and leaves the network).

We already know (see above) the power effects of a known current going through a resistor, capacitor or inductor, so we can just add up these powers and determine what's going on in terms of both the active power (the power dissipated in the resistor) and the reactive power (oscillating into and out from the capacitor or inductor), in terms of the current flowing.

30.6 Using phasors to calculate power

All seems reasonably straightforward so far, I hope. Things can, however, start to get a little confusing when phasor representations of the voltage and current are used.

Instantaneous power is determined by multiplying together the instantaneous voltage and the instantaneous current. However, phasors were never designed to be multiplied together: you can't determine a phasor representing the power by simply multiplying together the phasor representing the voltage and the phasor representing the current.

However, it turns out that there is a useful way to derive information about the power in an AC circuit from the phasor representing the voltage and the phasor representing the current. Before seeing how this can be done, it might be instructive to see what goes wrong if you do simply multiply the phasors together…

30.6.1 What doesn't work: just multiplying the phasors together

Consider a resistor, with a phasor of magnitude $|I_R|$ and phase θ representing the current through it. The voltage across this resistor would, in phasor terms, be calculated as:

$$\mathbf{V_R} = R\left|\mathbf{I_R}\right|\exp\left(j\theta\right) \tag{30.22}$$

where R is the (entirely real) impedance of the resistor. Multiplying the voltage phasor with the current phasor to try and produce some sort of "power phasor" would then give:

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[4] It's just a Thévenin equivalent network with no voltage source.

$$\mathbf{P_R} = \mathbf{V_R I_R} = R|\mathbf{I_R}|\exp(j\theta)|\mathbf{I_R}|\exp(j\theta) = R|\mathbf{I_R}|^2\exp(2j\theta) \tag{30.23}$$

and converting this back from phasor notation to a real quantity would give:

$$\begin{aligned} \Re\{\mathbf{P_R}\exp(j\omega t)\} &= \Re\left\{R|\mathbf{I_R}|^2\exp(2j\theta)\exp(j\omega t)\right\} \\ &= \Re\left\{R|\mathbf{I_R}|^2\exp(j(\omega t + 2\theta))\right\} \\ &= R|\mathbf{I_R}|^2\cos(\omega t + 2\theta) \end{aligned} \tag{30.24}$$

Hang on a minute… that can't be right. The average value of this "power" (in fact the average value of any signal represented by a phasor) is zero, but we know that there is real average power being lost in a resistor. So this can't be the power dissipated in the resistor. What's gone wrong?

The simple truth is you can't multiply two phasors to produce another phasor. It just doesn't work.

30.6.2 What does work: multiply the voltage phasor by the complex conjugate of the current phasor and then divide by two

Let's approach this problem from a different direction: multiplying two complex quantities together will in general produce a complex answer. What would we want a complex representation of power to look like?

From the calculations above (see equation (30.5)), it appears that the active power, expressed in terms of the complex impedance of a resistor (which is just *R*), is:

$$P(t) = \frac{V_0 I_0}{2} + \frac{V_0 I_0}{2}\cos(2\omega t) = \frac{I_0^{\,2}R}{2} + \frac{I_0^{\,2}R}{2}\cos(2\omega t) \tag{30.25}$$

and the reactive power, expressed in terms of the complex impedance of the capacitor or inductor, is (see equation (30.18)):

$$P(t) = j\frac{I_0^{\,2}Z}{2}\sin(2\omega t) \tag{30.26}$$

The two quantities of most interest are the average value of the active power (which is the average rate of energy actually being lost to the circuit and therefore available to do useful work) and the maximum value of the reactive power (the largest rate of energy oscillating from the source to the load and back). The first works out to be a real quantity when complex impedances are used; the second works out to be a purely imaginary quantity.

So, it would seem sensible to define a *complex power* to have a real part equal to average value of the active power, and an imaginary component equal to the maximum magnitude of the oscillating reactive power. Then we could use the Argand diagram to illustrate the complex, active and reactive powers, for example like this:

$$S = P + jQ = \frac{|I|^2}{2}(R + jX)$$

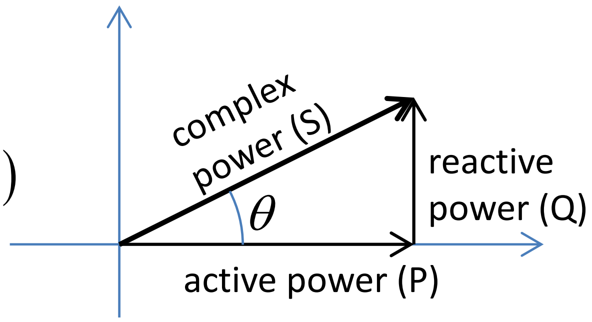

Figure 30.5 The complex power triangle on an Argand diagram

It turns out that there is an easy way to derive exactly this complex power from the phasors representing the voltage and the current. But you don't just multiply them: you multiply the voltage with the complex conjugate of the current and then divide by two.

$$\mathbf{S} = \frac{\mathbf{V} \times \mathbf{I}^*}{2} \tag{30.27}$$

where **S** is the complex power, and **V** and **I** are the phasor representations of the voltage and current respectively.

To see why this works, start by considering a complex representation of the impedance, plotted on an Argand diagram (see Figure 30.6). Let this complex impedance have a resistive component *R* and a reactive component *X*, so that:

$$Z = R + jX \tag{30.28}$$

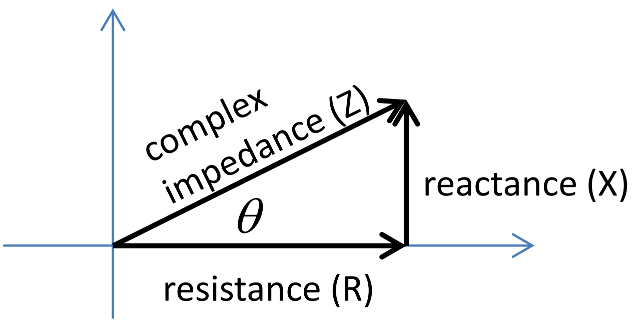

Figure 30.6 The complex impedance triangle: showing resistance and reactance

If we multiply the complex impedance[5] by the phasor representation of the current, we'd end up with a phasor representing the voltage across the component. Let this current phasor have a magnitude of |**I**| and a phase of θ, so this will produce a voltage phasor of:

$$\mathbf{V} = |\mathbf{I}|\exp(j\theta)(R + jX) \tag{30.29}$$

[5] which is not a phasor (however it is the ratio of two phasors: a voltage phasor and a current phasor).

If this voltage phasor is then multiplied not by the current phasor, but by the complex conjugate of the current phasor:

$$\mathbf{I}^* = |\mathbf{I}|\exp(-j\theta) \tag{30.30}$$

the result is:

$$\begin{aligned}\mathbf{V}\mathbf{I}^* &= |\mathbf{I}|\exp(j\theta)(R+jX)|\mathbf{I}|\exp(-j\theta)\\ &= |\mathbf{I}|^2(R+jX)\end{aligned} \tag{30.31}$$

and dividing this by two gives:

$$\mathbf{S} = \frac{\mathbf{V}\mathbf{I}^*}{2} = \frac{|\mathbf{I}|^2(R+jX)}{2} = \frac{|\mathbf{I}|^2 R}{2} + j\frac{|\mathbf{I}|^2 X}{2} \tag{30.32}$$

Comparing this with equations (30.6) and (30.18) and noting that the magnitude of the current phasor $|\mathbf{I}|$ is the same as the peak value of the real current I_0, we can write this as:

$$\mathbf{S} = \frac{\mathbf{V}\mathbf{I}^*}{2} = P + jQ \tag{30.33}$$

where *P* is the average active (or real) power, and *Q* is the maximum instantaneous reactive power.

Before leaving this topic, I should point out a couple of important consequences of this definition of complex power. Firstly, the magnitude of the complex power (known as the *apparent power* and usually measured in units of VA to emphasise that it's not a real power (which would be measured in watts)), is exactly what you would get if you measured the voltage with an rms voltmeter and the current with an rms ammeter and multiplied the readings together, since:

$$V_{rms}I_{rms} = \frac{|\mathbf{V}|}{\sqrt{2}}\frac{|\mathbf{I}|}{\sqrt{2}} = \frac{|\mathbf{V}||\mathbf{I}|}{2} = \frac{|\mathbf{V}\mathbf{I}^*|}{2} = |\mathbf{S}| \tag{30.34}$$

Secondly, the argument of the complex power is:

$$\arg(\mathbf{S}) = \tan^{-1}\left(\frac{I_0{}^2 X}{I_0{}^2 R}\right) = \tan^{-1}\left(\frac{X}{R}\right) \tag{30.35}$$

which is exactly the same as the phase angle of the complex impedance of the load network.

This second consequence becomes rather obvious when you consider where the complex power comes from: starting from a complex representation of the impedance, multiplying by the current phasor rotates the diagram by the phase angle of the current; but then multiplying the resultant voltage by the complex conjugate of the current rotates the diagram back through the same phase angle, resulting in a *power triangle* which is congruent with the complex impedance triangle we started with (see Figure 30.7 below).

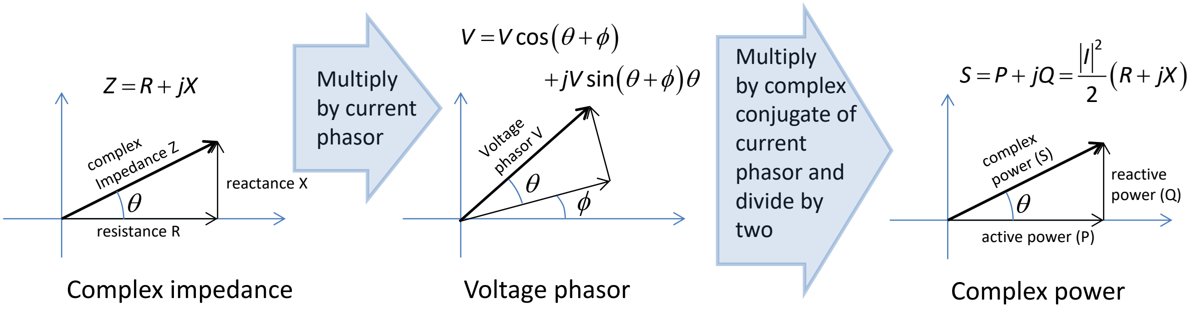

Figure 30.7 Figure showing congruence of impedance and power triangles

30.7 The power factor

As we've seen, any arbitrary two-terminal linear, passive network can be treated as having a complex impedance at any given frequency, and this complex impedance can be defined in terms of its real and imaginary components:

$$Z = R + jX \tag{30.36}$$

or in polar coordinates using Euler's result:

$$Z = \sqrt{R^2 + X^2}\exp\left(j\tan^{-1}\left(\frac{X}{R}\right)\right) \tag{30.37}$$

We've also noted that determining the complex power dissipated in this network results in a quantity with the same phase angle: $\tan^{-1}(X/R)$. The cosine of this angle is known as the *power factor*:

$$f = \cos\left(\tan^{-1}\left(\frac{X}{R}\right)\right) = \frac{R}{\sqrt{X^2 + R^2}} \tag{30.38}$$

From a consideration of the *power triangle* in Figure 30.5, this can also be identified as the ratio of the active power to the apparent power:

$$f = \frac{\text{active power}}{\text{apparent power}} \tag{30.39}$$

This is a convenient definition, since it means that an entirely real impedance which results in all of the apparent power leaving the circuit (for example in a resistor) would have a power factor of one, and an entirely reactive impedance which results in no power leaving the circuit (for example in a capacitor or inductor) would have a power factor of zero.

30.8 An example of complex, apparent, active and reactive power

For example, consider a load containing the series combination of a capacitor of value *C* farads and a resistor of *R* ohms, in a circuit like this:

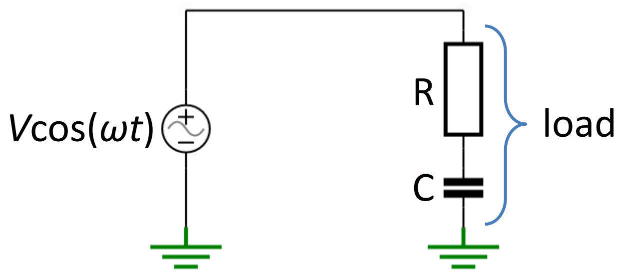

Figure 30.8 Example circuit for analysing active and reactive power

This circuit is then connected to a voltage source of magnitude *V* volts, angular frequency *ω* rad/s and phase zero. What is the active and reactive power at the load, and what maximum current flows between the source and the load?

The first task is to determine the complex impedance of the load. In this case it will be:

$$Z = R + \frac{1}{j\omega C} \tag{30.40}$$

We're not told the current here, so we have to work it out from the voltage and the complex impedance. The phasor representing the voltage will **V** be a real number (since the phase of the voltage source is zero), and the current phasor will be:

$$\mathbf{I} = \frac{\mathbf{V}}{Z} = \frac{\mathbf{V}}{R + \frac{1}{j\omega C}} = \frac{j\omega C\mathbf{V}}{1 + j\omega RC} \tag{30.41}$$

The complex power is obtained by multiplying the voltage phasor with the current phasor and dividing by two, which here gives:

$$S = \frac{\mathbf{V}\mathbf{I}^*}{2} = \frac{\mathbf{V}}{2}\left(\frac{-j\omega C\mathbf{V}^*}{1 - j\omega RC}\right) = \frac{-j\omega C|\mathbf{V}|^2}{2 - 2j\omega RC} \tag{30.42}$$

To determine the active and reactive powers, we need to find the real and imaginary parts of this complex power:

$$\begin{aligned}
\mathbf{S} &= \frac{-j\omega C|\mathbf{V}|^2}{2 - 2j\omega RC} \times \frac{2 + 2j\omega RC}{2 + 2j\omega RC} \\
&= \frac{-j\omega C|\mathbf{V}|^2 2(1 + j\omega RC)}{4 + 4\omega^2 R^2 C^2} \\
&= \frac{\omega C|\mathbf{V}|^2(-j + \omega RC)}{2(1 + \omega^2 R^2 C^2)} \\
&= \frac{\omega^2 RC^2|\mathbf{V}|^2}{2(1 + \omega^2 R^2 C^2)} - \frac{\omega C|\mathbf{V}|^2}{2(1 + \omega^2 R^2 C^2)}
\end{aligned} \tag{30.43}$$

so the active power P is:

$$P = |\mathbf{V}|^2 \frac{\omega^2 RC^2}{2\left(1+\omega^2 R^2 C^2\right)} \tag{30.44}$$

and the reactive power Q is:

$$Q = |\mathbf{V}|^2 \frac{-\omega C}{2\left(1+\omega^2 R^2 C^2\right)} \tag{30.45}$$

30.9 Power factor correction

While reactive power does not represent power actually lost to the circuit, it does increase the current required from the power source: reactive power in the load implies that additional power is being taken from the source only to be returned later in the cycle, which means that the power source has to be capable of supplying more instantaneous power. If the load is a long distance from the power source, then the effect of this additional current through the power cables connecting the source to the load can also increase the power lost due to resistance in the cables[6]. To minimise these losses, it helps to reduce the reactive power as much as possible which in turn implies reducing the reactive component of the load network's impedance.

Doing this is known as *power factor correction*. The idea is to add additional reactive components to the load so that it appears to have an impedance that is entirely real (and hence a power factor of one). This way there are no additional currents associated with the reactive power flow, and this minimises the additional power loss in the power delivery system.

For example, consider a motor specified as having a power requirement of 3 kVA[7], and a power factor of 0.8 (in other words the impedance of the motor has a phase angle of $\cos^{-1}(0.8)$ = 37 degrees). The motor is to be driven from a supply which has a zero-to-peak amplitude of 353 volts and a frequency of 50 Hz[8]. How much peak current is required?

We can draw an Argand diagram representing this situation as follows:

[6] Some energy suppliers charge more per watt for customers who don't have a good power factor.

[7] The use of kVA units in the motor's power specification implies that this is an apparent power rating, not the active power rating, which would be specified in watts. This is a common way to specify motors.

[8] 353 V zero-to-peak is equivalent to 240 V rms, and is the average level of the domestic mains power supply in the UK.

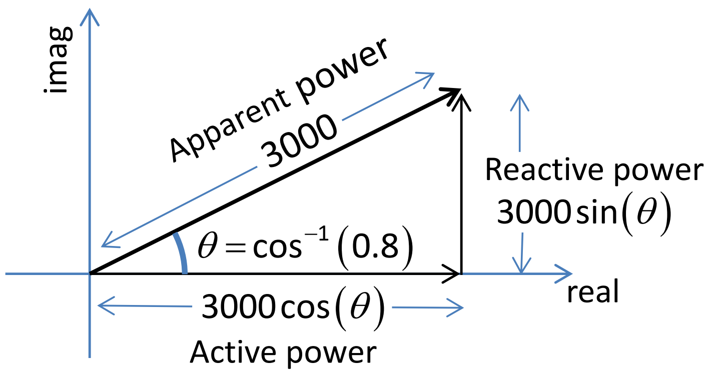

Figure 30.9 Diagram showing power triangle for the 3 kVA motor

The apparent power is 3 kVA, and the phasor representing the voltage would have a magnitude of 353 volts, so we can determine the magnitude of the current from:

$$\begin{aligned} |\mathbf{S}| &= \frac{|\mathbf{V}\mathbf{I}^*|}{2} = \frac{|\mathbf{V}||\mathbf{I}^*|}{2} \\ 3000 &= \frac{353\,|\mathbf{I}|}{2} \\ |\mathbf{I}| &= \frac{2\times 3000}{353} = 17\,\mathrm{A} \end{aligned} \tag{30.46}$$

Although it might seem unlikely, it is possible to reduce the magnitude of this current without reducing the active power delivered to the resistive part of the load (which does the actual work of moving the motor).

Noting that the power factor for this motor is given as 0.8, and the apparent power is 3 kVA when supplied with an oscillating voltage with a magnitude of 353 V, using the result from equation (30.32) we can determine the complex impedance of the motor:

$$\begin{aligned} \mathbf{S} &= \frac{|\mathbf{I}|^2 (R + jX)}{2} \\ 2\times 3000\exp\left(j\cos^{-1}(0.8)\right) &= 17^2 (R + jX) \\ Z = R + jX &= \frac{6000\exp\left(j\cos^{-1}(0.8)\right)}{17^2} \end{aligned} \tag{30.47}$$

This has a real part of:

$$R = \frac{6000\times 0.8}{17^2} = 16.609\ \Omega \tag{30.48}$$

and an imaginary part of:

$$X = \frac{6000\sin\left(\cos^{-1}(0.8)\right)}{17^2} = 12.457\ \Omega \tag{30.49}$$

and it's this reactance which is causing the reactive power to flow back and forth from the source to the load (the motor). Now suppose we add a capacitor in parallel with the motor so that the total impedance of the motor and capacitor is entirely real.

The simplest way to calculate the capacitance required is to note that when components are placed in parallel, the inverse of the sum of the total impedance is the sum of the inverses of the individual impedances:

$$\frac{1}{Z_{total}} = \frac{1}{Z_1} + \frac{1}{Z_2} \tag{30.50}$$

and hence if the total impedance is to be entirely real, so must the sum of the inverses of the individual components.

The inverse of the motor's impedance is:

$$\frac{1}{Z_{motor}} = \frac{1}{R+jX} = \frac{1}{R+jX} \times \frac{R-jX}{R-jX} = \frac{R-jX}{R^2+X^2} = \frac{R}{R^2+X^2} - j\frac{X}{R^2+X^2} \tag{30.51}$$

So it the total is to be entirely real, then the inverse of the impedance of the capacitor must cancel out the imaginary part of the inverse of the motor's impedance. This gives:

$$\frac{1}{Z_{cap}} = j\omega C = j\frac{X}{R^2+X^2} \tag{30.52}$$

from which we can immediately see that we need a capacitor of capacitance:

$$C = \frac{X}{\omega\left(R^2+X^2\right)} \tag{30.53}$$

(and just to check, this makes the total impedance of the network:

$$\begin{aligned} \frac{1}{Z_{total}} &= \frac{R}{R^2+X^2} - j\frac{X}{R^2+X^2} + j\frac{X}{R^2+X^2} = \frac{R}{R^2+X^2} \\ Z_{total} &= \frac{R^2+X^2}{R} \end{aligned} \tag{30.54}$$

which is entirely real as required).

Putting in some numbers, the capacitor required would be:

$$\begin{aligned} C &= \frac{X}{\omega\left(R^2+X^2\right)} = \frac{12.457}{2\pi 50\left(12.457^2+16.609^2\right)} \\ &= \frac{12.457}{135413} = 92\,\mu\text{F} \end{aligned} \tag{30.55}$$

and the maximum current taken by the network motor would now be:

$$I = \frac{V}{Z_{total}} = \frac{353}{16.609 + \frac{12.457^2}{16.609}} = 13.6 \text{ A} \quad (30.56)$$

which is around 3.4 amps less than before. This reduced current flow from the voltage source can reduce the losses in the power cables, which gives a particularly large improvement if the motor is some distance from the power source.

Nothing has really changed as far as the motor is concerned the same current is flowing through it, and the same voltage appears across it. All that's happened is that the energy that is being alternately stored and released from the inductive components of the motor's complex impedance is now being supplied and then returned to a capacitor placed right next to the motor, rather than being supplied and returned to the power source which could be some distance away.

(You might be wondering why the capacitor is not added in series with the motor, rather than in parallel with it. After all, that way you could directly cancel out the inductive component of the motor's impedance.

One problem is that in this configuration the capacitor would have to have a reactance of –*j* 12.457 ohms, which means it requires a value of:

$$C = \frac{1}{j\omega Z_C} = \frac{1}{j\omega(-j12.457)} = \frac{1}{2\pi 50 \times 12.457} = 255 \ \mu\text{F} \quad (30.57)$$

which is substantially larger, and therefore more expensive to provide. This is perhaps not surprising, since in this case the capacitor has all of the motor's current flowing through it, not just the reactive current being reflected back from the motor's inductance.

Another problem with this idea is that adding a capacitor in series in this way would change the operation of the motor. It would reduce the overall impedance of the load, which would increase the current, and hence the voltage across the motor.

For example, in this case, the overall impedance of the load when compensated with a parallel compensation capacitor is:

$$Z_{total} = \frac{R^2 + X^2}{R} = \frac{16.609^2 + 12.457^2}{16.609} = 25.95 \ \Omega \quad (30.58)$$

whereas the overall impedance of the load when compensated with a series compensation capacitor would be:

$$Z_{total} = R + X - X = 16.609 \ \Omega \quad (30.59)$$

The current flowing through the circuit in this case would therefore be:

$$I_{circuit} = \frac{353}{16.609} = 21.25 \text{ A} \quad (30.60)$$

and this greater current would result in a voltage across the motor of:

$$\begin{aligned}\left|V_{motor}\right| &= I_{circuit}\left|Z_{motor}\right| = 21.25\times\left|16.609 + j12.457\right| \\ &= \left|353 + j264\right| \\ &= 441\text{ V}\end{aligned} \tag{30.61}$$

The voltage across the motor itself has increased, due to the higher current flowing. If the motor was only rated to have mains voltage levels across it, the motor could burn out. This is perhaps the more serious problem of the two, and the main reason why series compensation is rare for power devices such as motors.)

30.10 Summary: the most important things to know

- The rms power dissipated across a resistor is given by $|\mathbf{v}|^2 / 2R$ where **v** is the phasor representing the voltage across the resistor.
- No power is dissipated in capacitors or inductors, however they alternately store and return energy to the circuit.
    - The maximum rate at which these reactive components store energy is given by $|\mathbf{v}\,\mathbf{i}|^2 / 2$ where **v** is the phasor representing the voltage across the component and **i** is the phasor representing the current flowing through it.
    - The energy is stored in the electric fields in capacitors and the magnetic field in inductors.
- Complex power of a component is given by $\mathbf{S} = \mathbf{v}\,\mathbf{i}^* / 2$
    - The magnitude of the complex power is the apparent power, measured in VA.
    - The real part of the complex power is the active power, measured in watts.
        - The active power is the power dissipated in the component.
    - The imaginary part of the complex power is the reactive power, measured in var.
        - The reactive power is the maximum rate of energy storage or release.
    - The cosine of the argument of the complex power is the power factor.
        - The argument of the complex power is equal to the argument of the complex impedance.
- The current from the source to a load can be minimised using power factor compensation.
    - Power factor compensation involves putting another reactive component in parallel with the load in order to make the load impedance look entirely real.
- To dissipate the maximum power in the load, the impedance of the load should be the complex conjugate of the Thévenin impedance of the source.