26 A Short Introduction to Second-Order Step Responses

v1.7 – June 2021

Prerequisite knowledge required: First-order responses, second-order frequency responses

26.1 Introduction

As we've seen in the "Second Order Frequency Response" chapter, there are a large number of different possible second-order responses, too many to consider them all. So again, I'll just concentrate on a few of the more interesting cases for the sake of saving time and space.

I'll also refrain from doing a step-by-step general solution of the second-order differential equations which result from the analysis of the circuits, instead taking the approach of making an intelligent guess about the form of the solution, and then testing to see if it works. This saves a lot of time.

Since it's a second-order response, there will now be two constants of integration introduced when the differential equation is solved, so we'll need two different, known values of the step response to establish these constants. Unfortunately the initial value theorem and the final value theorem are often not enough, and we have to look at the circuits more carefully to complete the solutions.

Right… on with the analysis.

26.2 A second-order low-pass filter with no zeros

This is the first example studied in the chapter on second-order frequency responses. It's the output from the circuit shown below:

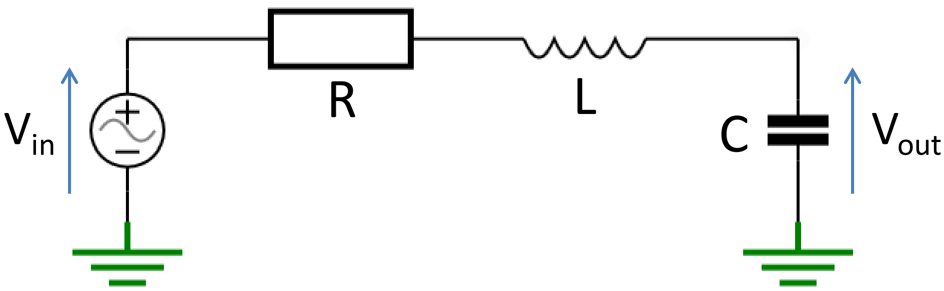

Figure 26.1 A series RLC circuit with the output taken across the capacitor

A nodal analysis of this circuit reveals that:

$$\begin{aligned} V_{in}(t) - V_X(t) &= I_R(t) \times R \\ V_X(t) - V_{out}(t) &= L\frac{dI_L(t)}{dt} \\ I_C(t) = \frac{dQ(t)}{dt} &= C\frac{dV_{out}(t)}{dt} \\ I_R(t) = I_C(t) &= I_L(t) \end{aligned} \tag{26.1}$$

where V_X is the voltage at the node marked 'X' in the circuit diagram (between the resistor and the capacitor), and I_R, I_L and I_C are the currents through the resistor, inductor and capacitor respectively (all taken as positive when the current is going clockwise).

Some straight-forward algebra to eliminate V_X and the three currents from these equations gives the second-order differential equation:

$$LC\frac{d^2V_{out}(t)}{dt^2}+RC\frac{dV_{out}(t)}{dt}+V_{out}(t)=V_{in}(t) \tag{26.2}$$

For a step input, after time $t = 0$ (which is the only time we're interested in), V_{in} is a constant value of one, so we can write:

$$LC\frac{d^2V_{out}(t)}{dt^2}+RC\frac{dV_{out}(t)}{dt}+V_{out}(t)=1 \tag{26.3}$$

Now the intelligent guess: assume that there is a solution of the form:

$$V_{out}(t)=A\exp(\alpha t)+B\exp(\beta t)+C \tag{26.4}$$

This would then give:

$$\begin{aligned}\frac{dV_{out}(t)}{dt}&=\alpha A\exp(\alpha t)+\beta B\exp(\beta t)\\ \frac{d^2V_{out}(t)}{dt^2}&=\alpha^2 A\exp(\alpha t)+\beta^2 B\exp(\beta t)\end{aligned} \tag{26.5}$$

and substituting this into equation (26.3) gives:

$$\begin{gathered}LC\alpha^2 A\exp(\alpha t)+LC\beta^2 B\exp(\beta t)+RC\alpha A\exp(\alpha t)RC\beta B\exp(\beta t)\\ +A\exp(\alpha t)+B\exp(\beta t)+C=1\end{gathered} \tag{26.6}$$

For this to work for all times, the coefficients of the two exponential decays must be equal on both sides, and for terms in exp(αt) this suggests we need:

$$LC\alpha^2 A+RC\alpha A+A=0 \tag{26.7}$$

Dividing through by A then gives the formula:

$$LC\alpha^2+RC\alpha+1=0 \tag{26.8}$$

Doing the same for the coefficients of exp(βt) gives exactly the same formula, except with β instead of α.

We also need the two sides to be equal at infinite time, when the terms in exp(αt) and exp(βt) are zero, and this gives:

$$C=1 \tag{26.9}$$

At this point we know the value of C, and we have a quadratic equation to determine the values of α and β. That's convenient, since quadratic equations have two solutions, and we can use one for α and the other for β. Solving this quadratic gives:

$$\alpha,\beta = -\frac{RC \pm \sqrt{R^2C^2 - 4LC}}{2LC} = -\frac{RC}{2LC}\left(1 \pm \sqrt{1 - \frac{4L}{R^2C}}\right) \tag{26.10}$$

I've left the expression in this form for now (rather than dividing through by *C*) because it's useful to note that for this second-order response $LC = 1/\omega_0^2$, $L/R^2C = Q^2$, and $RC = 1/Q\omega_0$, so we can write equation (26.10) here in terms of the resonant frequency and Q-factor as:

$$\alpha,\beta = -\frac{\omega_0^{\;2}}{2Q\omega_0}\left(1 \pm \sqrt{1 - 4Q^2}\right) = -\frac{\omega_0}{2Q}\left(1 \pm \sqrt{1 - 4Q^2}\right) \tag{26.11}$$

At this point it's useful to note that the poles in this system are (see the chapter on second-order frequency responses for the derivation):

$$p_{0,1} = -\omega_0\left(\frac{1 \pm \sqrt{1 - 4Q^2}}{2Q}\right) \tag{26.12}$$

and by comparing equations (26.11) and (26.12) it's immediately apparent that the values of α and β are just the two poles of the frequency response. So I'll write them as p_0 and p_1 from now on.

The next problem is how to work out the values of *A* and *B*.

The initial value theorem provides one equation: at time $t = 0$ we know that the output is zero since the capacitor cannot suddenly change voltage, so we can write:

$$\begin{gathered} A\exp(p_0 \times 0) + B\exp(p_1 \times 0) + C = A + B + C = 0 \\ A + B = -1 \end{gathered} \tag{26.13}$$

but we need another equation. The final value theorem can't help, this would just give that:

$$A\exp(p_0\,\infty) + B\exp(p_1\,\infty) + C = 0 + 0 + C = 1 \tag{26.14}$$

and we already know that. To get the final equation we need, we have to go back to the original circuit equations, and in particular look at the rate of change of the output voltage with time. First, note that for the inductor:

$$V_X(t) - V_Y(t) = L\frac{dI_L(t)}{dt} \tag{26.15}$$

You can't suddenly change the current through an inductor, since that would require an infinite value of dI/dt, and that implies an infinite voltage at X or Y, which is impossible. So the current just after $t = 0$ must be the same as the current just before $t = 0$, which is to say it must be zero.

Now, for the capacitor, we have:

$$I_C(t) = \frac{dQ(t)}{dt} = C\frac{dV_{out}(t)}{dt} \tag{26.16}$$

and from this we can deduce that if the current just after $t = 0$ is zero, then the rate of change of V_{out} with time (dV_{out}/dt) must also be zero just after $t = 0$.

Hence from equation (26.5) we have that just after $t = 0$:

$$\frac{dV_{out}(t)}{dt} = p_0 A\exp(0) + p_1 B\exp(0) = 0 \tag{26.17}$$

which gives the other equation we need:

$$p_0 A + p_1 B = 0 \tag{26.18}$$

This, together with equation (26.13) provide two simultaneous equations in A and B that can then be solved to give:

$$A = \frac{p_1}{p_0 - p_1} \qquad B = \frac{p_0}{p_1 - p_0} \tag{26.19}$$

which gives the final solution as:

$$V_{out}(t) = 1 + \frac{p_1}{p_0 - p_1}\exp(p_0 t) + \frac{p_0}{p_1 - p_0}\exp(p_1 t) \tag{26.20}$$

where:

$$p_0 = -\omega_0\left(\frac{1+\sqrt{1-4Q^2}}{2Q}\right) \qquad p_1 = -\omega_0\left(\frac{1-\sqrt{1-4Q^2}}{2Q}\right) \tag{26.21}$$

There are three cases of interest here: the case where $Q < 0.5$ (and hence both poles are real), the case where $Q > 0.5$ (where both poles are complex) and the annoyingly awkward case of $Q = 0.5$, which has two equal poles. This last one requires more study: see the next section for why.

26.2.1 The awkward case of Q = 0.5

This is interesting, since if $Q = 0.5$ then both poles coincide and $p_0 = p_1$. This means solving equation (26.20) involves dividing by zero. That's not going to work: something is wrong.

To solve this one, we have to go back to the original assumption:

$$V_{out}(t) = A\exp(\alpha t) + B\exp(\beta t) + C \tag{26.22}$$

and have another go at an intelligent guess for a solution. In this situation, the one that works is to assume that there is a solution of the form:

$$V_{out}(t) = A\exp(\alpha t) + Bt\exp(\alpha t) + C \tag{26.23}$$

and follow through to find suitable values for A, B, C and α. The derivation follows the same steps as outlined above. (I'll leave this as an exercise for the interested reader with time on their hands.) The eventual result is:

$$V_{out}(t) = 1 - (1 - p_0 t)\exp(p_0 t) \tag{26.24}$$

and this looks like this:

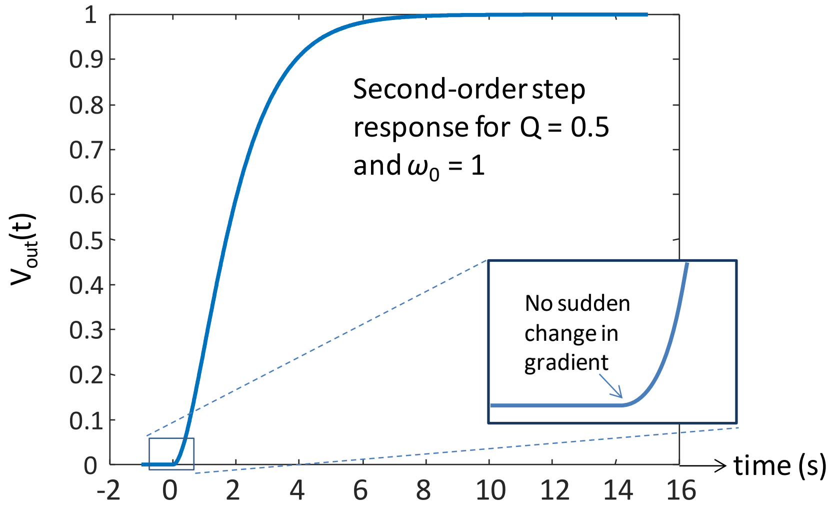

Figure 26.2 Second-order low-pass filter step response with Q = 0.5

Note in particular that the gradient doesn't suddenly change at time t = 0 (as it does in the case of the first-order response). This is characteristic of second-order low-pass step responses and is often the easiest way to tell the difference between a first-order and second-order low-pass step response.

26.2.2 The case of Q < 0.5

When Q is significantly less than one-half, the two break frequencies of the two poles are both real and at very different frequencies. In these cases we often don't have to bother with all the maths above, we can often produce a good approximation to the solution using a technique called the *dominant pole approximation*.

26.2.2.1 The dominant pole approximation

The idea here is that when there are two poles widely spaced in frequency, the lower break frequency will dominate the frequency response. There are few different ways of thinking about why this might be true.

The first is to look at the general solution for this case (derived as equation (26.20) above):

$$V_{out}(t) = 1 + \frac{p_1}{p_0 - p_1}\exp(p_0 t) + \frac{p_0}{p_1 - p_0}\exp(p_1 t) \tag{26.25}$$

If p_0 is very much smaller than p_1, then the third term isn't going to have a significant effect at any point (since p_0 is so small it will never have a large value) and therefore we can neglect it. Further, in the second term we can approximate:

$$\frac{p_1}{p_0 - p_1} \approx \frac{p_1}{-p_1} = -1 \tag{26.26}$$

which leaves:

$$V_{out}(t) = 1 - \exp(p_0 t) \tag{26.27}$$

This, not surprisingly (since we've effectively just ignored one of the poles) has the same form as a first-order response with a single pole at p_0.

If the two pole break frequencies are more than a factor of around five apart (which corresponds to a Q factor of less than √5/6 = 0.3727[1]) then this gives a reasonable approximation of the overall response:

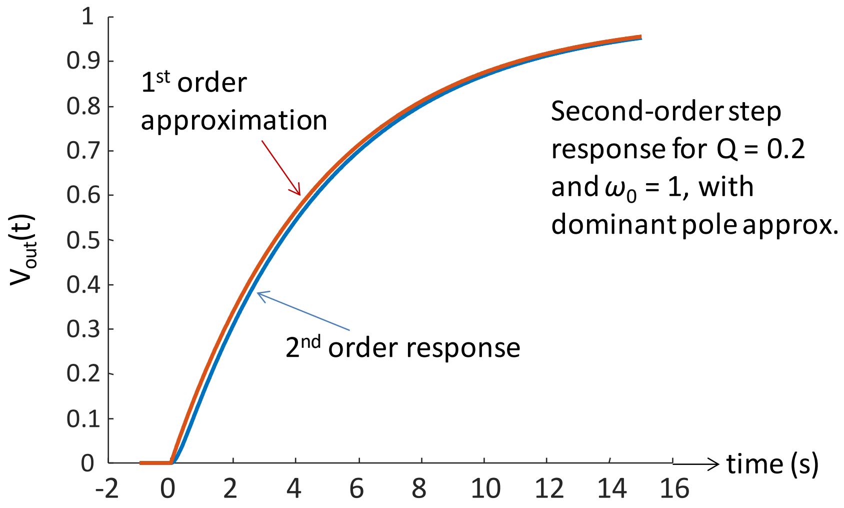

Figure 26.3 Second-order low-pass step response for Q = 0.2 with dominant pole approximation

For values of Q greater than 0.37 but still less than 0.5, the response deviates more noticeably from the first-order response: perhaps the most obvious difference is the fact that the second order response starts with the slope dV_{out}/dt being zero starts to become more obvious.

26.2.3 The case of Q > 0.5

The derivation of the step response in this case (where the poles are a complex) is a bit involved but it does lead to some important results. We start by noting that the two poles in the frequency response are a complex conjugate pair at:

$$p_{0,1} = -\omega_0 \left(\frac{1}{2Q} \pm j \frac{\sqrt{4Q^2 - 1}}{2Q} \right) \tag{26.28}$$

which expressed in polar terms[2], becomes:

$$p_{0,1} = -\omega_0 \exp\left(\pm j \tan^{-1}\left(\sqrt{4Q^2 - 1} \right) \right) \tag{26.29}$$

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[1] Deriving this result is a good exercise. You could also try extending it and showing that if the poles are both real and a factor X apart, then the Q-factor is $Q = \sqrt{X} / (X+1)$.

[2] Another good exercise is to derive equation (26.29) from equation (26.28). Try drawing out the location of the poles, and applying trigonometry and Pythagoras's theorem.

which suggests that:

$$\begin{aligned}\exp\left(p_{0,1}t\right) &= \exp\left(-\omega_0\left(\frac{1}{2Q}\pm j\frac{\sqrt{4Q^2-1}}{2Q}\right)t\right)\\ &= \exp\left(-\frac{\omega_0 t}{2Q}\right)\exp\left(\mp j\omega_0\frac{\sqrt{4Q^2-1}}{2Q}t\right)\end{aligned} \tag{26.30}$$

This gives the difference between the poles $p_0 - p_1$ as:

$$p_1 - p_0 = \omega_0\left(\frac{1}{2Q}+j\frac{\sqrt{4Q^2-1}}{2Q}\right)-\omega_0\left(\frac{1}{2Q}-j\frac{\sqrt{4Q^2-1}}{2Q}\right)=j\omega_0\frac{\sqrt{4Q^2-1}}{Q} \tag{26.31}$$

and substituting these results into equation (26.20) gives the general solution:

$$\begin{aligned}V_{out}\left(t\right) = 1 &+ \frac{\omega_0\exp\left(-j\tan^{-1}\left(\sqrt{4Q^2-1}\right)\right)}{j\omega_0\frac{\sqrt{4Q^2-1}}{Q}}\exp\left(-\frac{\omega_0 t}{2Q}\right)\exp\left(-j\omega_0\frac{\sqrt{4Q^2-1}}{2Q}t\right)\\ &- \frac{\omega_0\exp\left(+j\tan^{-1}\left(\sqrt{4Q^2-1}\right)\right)}{j\omega_0\frac{\sqrt{4Q^2-1}}{Q}}\exp\left(-\frac{\omega_0 t}{2Q}\right)\exp\left(+j\omega_0\frac{\sqrt{4Q^2-1}}{2Q}t\right)\end{aligned} \tag{26.32}$$

which simplifies to:

$$V_{out}\left(t\right) = 1 + \frac{\omega_0\exp\left(-\frac{\omega_0 t}{2Q}\right)}{j\omega_0\frac{\sqrt{4Q^2-1}}{Q}}\left(\begin{aligned}&\exp\left(-j\tan^{-1}\left(\sqrt{4Q^2-1}\right)-j\omega_0\frac{\sqrt{4Q^2-1}}{2Q}t\right)\\ &\qquad -\exp\left(j\tan^{-1}\left(\sqrt{4Q^2-1}\right)+j\omega_0\frac{\sqrt{4Q^2-1}}{2Q}t\right)\end{aligned}\right) \tag{26.33}$$

At this point it’s useful to remember that:

$$\sin\left(z\right) = \frac{e^{jz}-e^{-jz}}{2j} \tag{26.34}$$

which allows equation (26.33) to be rewritten in the much simpler form:

$$V_{out}\left(t\right) = 1 - \frac{2Q\exp\left(-\frac{\omega_0 t}{2Q}\right)}{\sqrt{4Q^2-1}}\sin\left(\omega_0\frac{\sqrt{4Q^2-1}}{2Q}t+\tan^{-1}\left(\sqrt{4Q^2-1}\right)\right) \tag{26.35}$$

and this has the form of a decaying oscillating response. For a couple of representative values of Q, the step response looks like this:

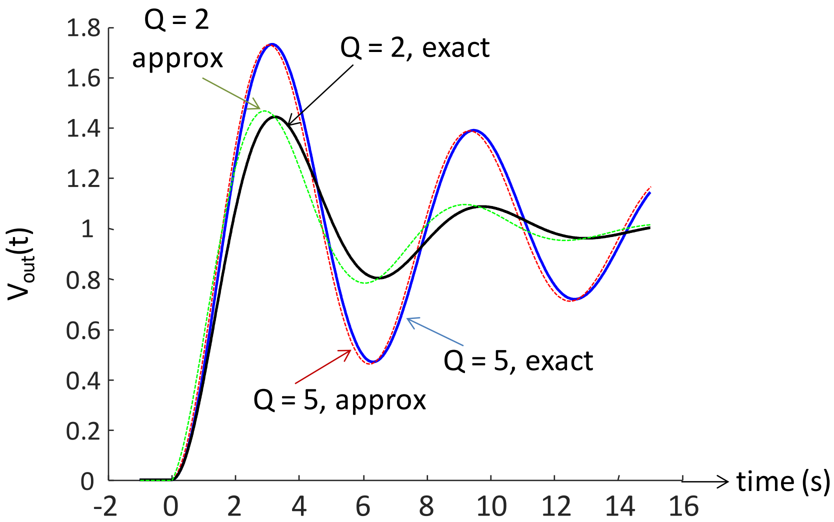

Figure 26.4 Second-order low-pass step response for Q = 2 and 5

(Note it's rather easier to derive this result by first assuming that the answer can be expressed in the form:

$$v_{out}(t) = A\sin(\omega t + \theta) + B \tag{26.36}$$

and substituting in to find *A*, *B*, *ω* and θ, but I'll leave that as an exercise as well; here I wanted to show how the same general solution can be used for complex poles as for real poles.)

From equation (26.35) we can derive that the frequency of the ringing is given by:

$$\omega_{ringing} = \omega_0 \frac{\sqrt{4Q^2 - 1}}{2Q} \tag{26.37}$$

which for large values of *Q* is approximately equal to the resonant frequency ω_0.

We can also derive that the initial amplitude of the ringing is given by:

$$A_{ringing} = \frac{2Q}{\sqrt{4Q^2 - 1}} \tag{26.38}$$

which for large values of *Q* is approximately equal to one, and that the decay of the oscillation has a time-constant of:

$$T_{ringing} = \frac{2Q}{\omega_0} \tag{26.39}$$

For large values of *Q* we can also approximate:

$$\tan^{-1}\left(\sqrt{4Q^2 - 1}\right) \approx \tan^{-1}(2Q) \approx \frac{\pi}{2} \tag{26.40}$$

and hence derive that for large values of *Q*, the step response can be well approximated by:

$$V_{out}(t) = 1 - \exp\left(-\frac{\omega_0 t}{2Q}\right)\sin\left(\omega_0 t + \frac{\pi}{2}\right) = 1 - \exp\left(-\frac{\omega_0 t}{2Q}\right)\cos(\omega_0 t) \qquad (26.41)$$

which is an oscillation at the resonant frequency which decays with the time-constant $2Q/\omega_0$.

26.2.3.1 Overshoot

This sort of response (with overshoot) is very common in practice and can cause problems in real circuits. A typical situation would be a logic gate (with a Thévenin equivalent output stage) driving a wire (which can be modelled as an inductor) into the input of another logic gate (which has a significant capacitance to ground).

For logic-gates powered from 3.3V (so that a logic zero is 0V and a logic one is 3.3V) this can result in an input to the second logic gate which looks like this (note all times here are normalised so that the angular frequency of the resonant frequency is 1 radian/second):

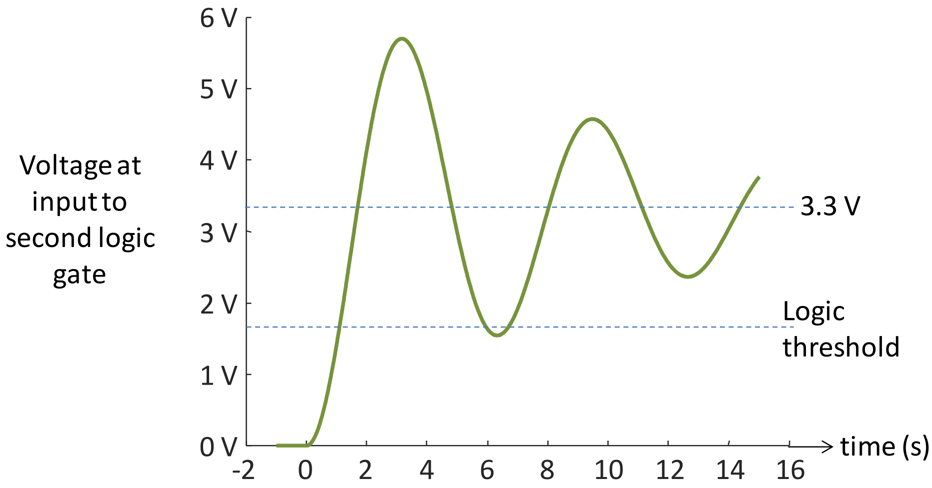

Figure 26.5 Input to logic gate driven by circuit with Q = 5 with normalised time axis

There are two issues here: the first is that if this input is a clock input, then it will clocked twice by the transition from a logic high at t = 0: once at around t = 1 and then again around at t = 6.5 due to the large ringing in the circuit.

The second issue is that the overshoot can exceed the maximum input voltage specifications for the second logic gate, and this can cause damage. It's therefore useful to be able to determine the maximum voltage achieved during the step response.

For large values of Q, this can be determined by looking for turning points in the general solution. Differentiating equation (26.25) and setting the result to zero gives:

$$\frac{dx(t)}{dt} = \frac{p_0 p_1}{p_1 - p_0}\exp(p_1 t) + \frac{p_1 p_0}{p_0 - p_1}\exp(p_0 t) = 0 \qquad (26.42)$$

which suggests that turning points occur when:

$$\exp(p_1 t) = \exp(p_0 t) \qquad (26.43)$$

When the poles are complex, this equation has solutions other than at t = 0, since we have here:

$$p_{0,1} = \frac{-\omega_0}{2Q} \pm j\omega_0 \frac{\sqrt{4Q^2-1}}{2Q} \tag{26.44}$$

so the points where the gradient of the step-response is zero are given by:

$$\begin{aligned}
\exp\left(\frac{-\omega_0}{2Q}t + j\omega_0\frac{\sqrt{4Q^2-1}}{2Q}t\right) &= \exp\left(\frac{-\omega_0}{2Q}t - j\omega_0\frac{\sqrt{4Q^2-1}}{2Q}t\right)\\
\exp\left(\frac{-\omega_0}{2Q}t\right)\exp\left(j\omega_0\frac{\sqrt{4Q^2-1}}{2Q}t\right) &= \exp\left(\frac{-\omega_0}{2Q}t\right)\exp\left(-j\omega_0\frac{\sqrt{4Q^2-1}}{2Q}t\right)\\
\exp\left(j\omega_0\frac{\sqrt{4Q^2-1}}{2Q}t\right) &= \exp\left(-j\omega_0\frac{\sqrt{4Q^2-1}}{2Q}t\right)\\
\exp\left(2j\omega_0\frac{\sqrt{4Q^2-1}}{2Q}t\right) &= 1
\end{aligned} \tag{26.45}$$

The first solution to this equation (at t = 0) isn't interesting, since that is the start of the step response. The maximum value will occur at the second solution, which occurs where the argument has reached $j2\pi$ (since $\exp(j2\pi) = 1$). Here:

$$\begin{aligned}
2j\omega_0\frac{\sqrt{4Q^2-1}}{2Q}t &= j2\pi\\
t &= \frac{\pi}{\omega_0\frac{\sqrt{4Q^2-1}}{2Q}} = \frac{2Q\pi}{\omega_0\sqrt{4Q^2-1}}
\end{aligned} \tag{26.46}$$

Substituting back into equation (26.35) gives:

$$\begin{aligned}
V_{out}(t) &= 1 - \frac{2Q\exp\left(-\frac{\pi}{\sqrt{4Q^2-1}}\right)}{\sqrt{4Q^2-1}}\sin\left(\pi + \tan^{-1}\left(\sqrt{4Q^2-1}\right)\right)\\
&= 1 + \exp\left(-\frac{\pi}{\sqrt{4Q^2-1}}\right)
\end{aligned} \tag{26.47}$$

The maximum possible overshoot is 100% (an amplitude of 2), and this occurs when the Q-factor is infinite.

26.2.4 Summary of different step responses for the second-order low-pass filter

In summary there are three cases of particular interest:

- Q << 0.5 (two real poles with significantly different values): step response is similar to that of a first-order system with a pole frequency at the lower of the two poles in the second-order system

- Q = 0.5 (two co-incident poles): step response has a noticeable slower rise-time than the first-order case, especially as the gradient of V_{out} is clearly zero just after t = 0
- Q >> 0.5 (two complex poles): step response exhibits overshoot and ringing at a frequency close to the resonant frequency, the ringing decaying with a time-constant of $2Q / \omega_0$.

26.3 The second-order bandpass filter

The approach for the second-order bandpass filter is similar, and equally time-consuming if you start from scratch. However, if you have already derived the result for the low-pass filter (see above), then there's a short-cut.

A second-order with a bandpass response is shown below in Figure 26.6.

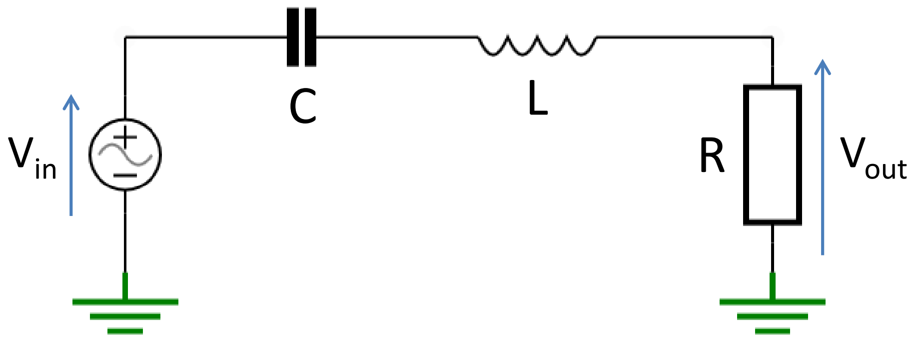

Figure 26.6 A second-order series RLC bandpass circuit

The only difference between this circuit and the low-pass filter analysed before is that the components are in a different order. This, however, doesn't affect the total impedance in the circuit (which is just the sum of the impedances of the capacitor, inductor and resistor), and therefore won't affect the total current flowing in the circuit. And if we know the current flowing in the circuit, all we have to do is multiply that current by the resistance R to get the voltage across the resistor, and this is the output voltage.

Now we know that for a capacitor:

$$I(t) = C\frac{dV}{dt} \tag{26.48}$$

and we know the voltage across the capacitor (that was the output in the low-pass filter case first analysed), so all we have to do is differentiate it, multiply by C to get the current, then multiply by R to get the voltage across the resistor. Substitute in for RC noting that the sum of the poles is equal to –RC in this circuit, and the result is:

$$V_{out}(t) = \frac{p_1 + p_0}{p_0 - p_1}\left(\exp(p_0 t) - \exp(p_1 t)\right) \tag{26.49}$$

except when the two poles are at the same frequency, when the output is:

$$V_{out}(t) = -2pt\,\exp(pt) \tag{26.50}$$

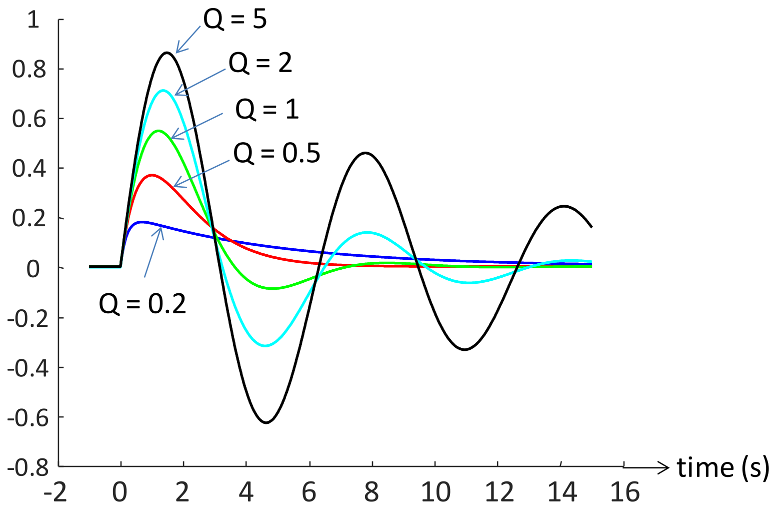

Figure 26.7 Step-response of the second-order bandpass filter

26.4 The second-order high-pass filter

If you take the output across the inductor instead of the resistor or the capacitor, then what results is a high-pass filter[3].

The circuit then looks like this:

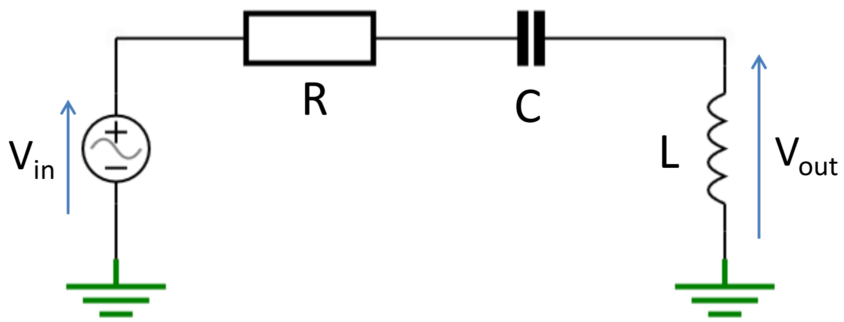

Figure 26.8 Second-order high-pass filter circuit

Again, the approach of solving this from scratch is tedious and left as an exercise to the reader, but again there is a short-cut: since we know the voltage across the capacitor (the low-pass case first analysed above), and we know the voltage across the resistor (the bandpass case analysed above), and we know that the sum of the voltages across the resistor capacitor and inductor must be one (Kirchhoff's voltage law), we can write:

$$V_L(t) = 1 - V_C(t) - V_R(t) \tag{26.51}$$

After some more tedious algebra, this leads to the output step response in this high-pass case:

[3] This is not as easy to do in practice since real inductors have significant parasitic resistance, which this time can't be added into the resistance of the resistor.

$$V_{out}(t) = \frac{1}{p_0 - p_1}\left(p_1 \exp(p_1 t) - p_0 \exp(p_0 t)\right) \tag{26.52}$$

except when the two poles are at the same frequency, when the output is:

$$V_{out}(t) = (1 + pt)\exp(pt) \tag{26.53}$$

The resultant step responses look like this:

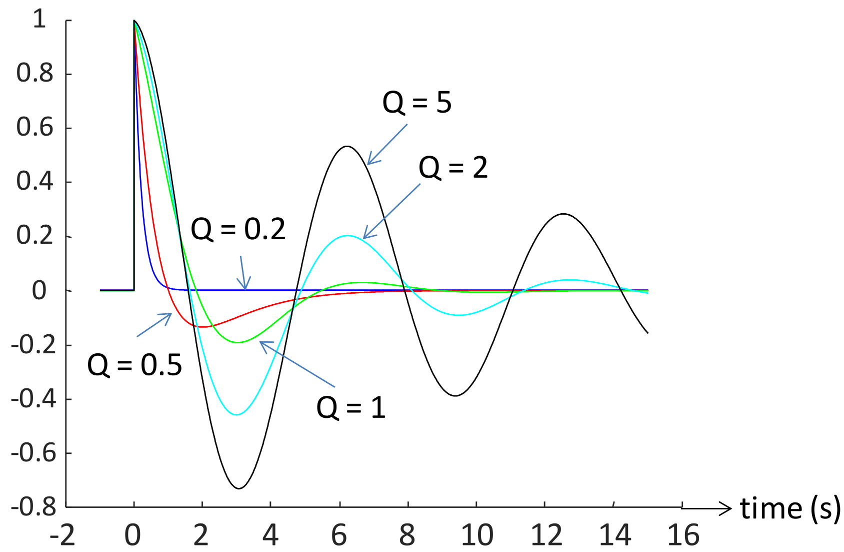

Figure 26.9 Step-response of second-order high-pass filter

Note that in this case of Q = 0.5 there is some overshoot, even through the poles are not complex.

26.5 Everything else

The circuit above either have no zeros (the low-pass filter), one zero at zero Hz (the bandpass filter case) or two zeros at zero Hz (the high-pass filter case). There are a large number of other variants where the circuits have zeros at non-zero frequencies, but I won't attempt a derivation of any of these step responses in this chapter.

Partly this is due to a lack of space, but mostly it's because there is another technique for deriving step responses from the frequency response which doesn't require solving any differential equations. The maths required is beyond the syllabus of this module, but it's really not worth spending too long deriving the results of these cases here when there's a much easier method coming up later.

(For those who can't wait, the maths required is called the Laplace Transform. See any good textbook on Engineering Mathematics for more details.)

26.6 Summary: the most important things to know

- Second-order step responses have two time-constants: minus one times the inverses of the poles in the frequency responses.
- Complex poles in the frequency response lead to ringing in the step response.