25 A Short Introduction to Second-Order Frequency Responses

v1.8 – July 2021

Prerequisite knowledge required: First-order responses, Frequency responses

25.1 Introduction

While first-order filters can provide some useful frequency responses, many more useful filter shapes (including bandpass filters which can attenuate low and high frequencies and only pass through a certain range of frequencies) can be build using second-order filters with second-order responses. This chapter introduces second-order responses, shows how they are usually characterised, and discusses some special cases of particular practical interest.

25.2 Specifying and classifying second-order responses

Second order responses are characterised by the presence of two poles in the frequency response, for example:

$$H(j\omega) = \frac{1}{\left(1 + \frac{j\omega}{-p_0}\right)\left(1 + \frac{j\omega}{-p_1}\right)} \tag{25.1}$$

However there is an equivalently (and often more useful) form of the frequency response used in these cases: we can specify the two poles in terms of quantities known as the *resonant frequency* (ω_0) and *Q-factor* (Q):

$$H(j\omega) = \frac{1}{1 + (j\omega)\frac{1}{Q\omega_0} + \frac{(j\omega)^2}{\omega_0{}^2}} \tag{25.2}$$

Equating these expressions reveals that the poles can be given in terms of the resonant frequency and Q-factor by:

$$p_{0,1} = -\omega_0 \left(\frac{1 \pm \sqrt{1 - 4Q^2}}{2Q} \right) \tag{25.3}$$

or alternatively going in the other direction, expressing the resonant frequency and Q-factor in terms of the poles gives:

$$\omega_0{}^2 = p_0\, p_1 \qquad Q = -\frac{\sqrt{p_0\, p_1}}{p_0 + p_1} = -\frac{\omega_0}{p_0 + p_1} \tag{25.4}$$

and since the break frequencies of the poles are the modulus of the value of the poles, this gives in terms of the break frequencies of the poles ω_{p0} and ω_{p1}:

$$\omega_0{}^2 = \omega_{p0}\, \omega_{p1} \qquad Q = \frac{\sqrt{\omega_{p0}\, \omega_{p1}}}{\omega_{p0} + \omega_{p1}} = \frac{\omega_0}{\omega_{p0} + \omega_{p1}} \tag{25.5}$$

This gives a simple way of understanding what the resonant frequency and Q-factor represent:

- The resonant frequency is the geometric mean of the break frequencies of the poles
- The Q-factor is the resonant frequency divided by the sum of the break frequencies of the poles. This is a measure of how far the poles are apart: if the poles are equal, then the Q-factor is 0.5.

As we'll see later, several important properties of the frequency and step responses of these circuits can be more easily described in terms of the resonant frequency and Q-factor than in terms of the break frequencies of the poles themselves.

25.2.1 Classifying second-order responses

With two poles there are three possible situations, all of which produce slightly different results:

1) Two real poles with different break frequencies
2) Two real poles with the same break frequency
3) A pair of complex poles

There can also be up to two zeros in a second-order response as well, and in this case there are eight possibilities for the arrangement of the zeros:

1) No zeros in the response at all
2) One zero with break frequency of zero Hz
3) One zero with a non-zero break frequency
4) Two zeros, both with break frequencies at zero Hz
5) Two zeros, only one of which has a break frequency at zero Hz
6) Two real zeros with the same non-zero break frequency
7) Two real zeros with different break frequencies, neither at 0 Hz
8) A pair of complex zeros

This makes a total of twenty-four different cases to consider for the second-order response (not including the different possible orders of the poles and zeros where there is at least one zero at a non-zero frequency). It would take a book to go through all of them, so I'll just pick two example circuits to show the analysis method.

25.3 A series RLC circuit with the capacitor voltage as the output

The first example is perhaps the simplest: two poles with no zeros in the frequency response at all. A circuit that produces this response is the series-RLC circuit, when the output is taken as the voltage across the capacitor.

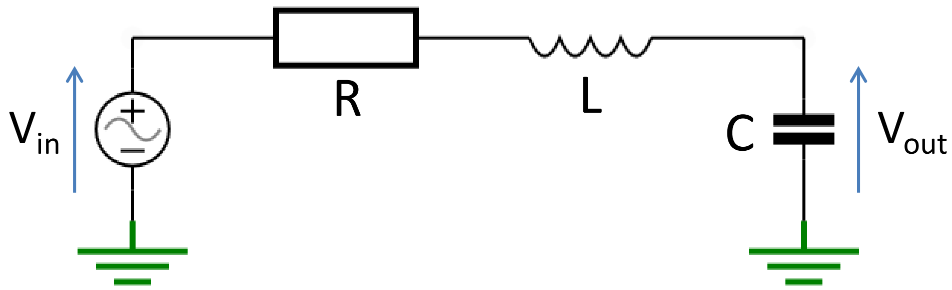

Figure 25.1 A series RLC circuit with the output taken across the capacitor

This circuit can be treated as a potential divider, which reveals the frequency response to be:

$$H(j\omega)=\frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}}=\frac{1/j\omega C}{R+j\omega L+1/j\omega C}=\frac{1}{1+j\omega RC+(j\omega)^2 LC} \tag{25.6}$$

and from comparing the denominator to the general form expressed in terms of the resonant frequency and the Q-factor:

$$1+j\omega RC+(j\omega)^2 LC=1+(j\omega)\frac{1}{Q\omega_0}+\frac{(j\omega)^2}{\omega_0{}^2} \tag{25.7}$$

it's not difficult to derive that:

$$\omega_0=\frac{1}{\sqrt{LC}} \qquad Q=\frac{1}{R}\sqrt{\frac{L}{C}} \tag{25.8}$$

At this point it's interesting to stop and consider the impedance of the capacitor and the inductor at the resonant frequency. The impedance of the inductor would be:

$$j\omega_0 L=j\frac{1}{\sqrt{LC}}L=j\sqrt{\frac{L}{C}} \tag{25.9}$$

and the impedance of the capacitor would be:

$$\frac{1}{j\omega_0 C}=-j\frac{\sqrt{LC}}{C}=-j\sqrt{\frac{L}{C}} \tag{25.10}$$

Notice that these are equal and opposite. At the resonant frequency, the impedance of the inductor and the capacitor cancel out, and the only impedance left in the circuit is that of the resistor. This means that the current flowing around the circuit (expressed as a phasor) would be:

$$\mathbf{I}=\frac{\mathbf{V}_{\text{in}}}{R} \tag{25.11}$$

and the voltage generated across the capacitor (again as a phasor) would therefore be:

$$\mathbf{V}_{\text{out}}=\mathbf{I}\times\frac{1}{j\omega_0 C}=\frac{\mathbf{V}_{\text{in}}\sqrt{LC}}{jCR}=-j\mathbf{V}_{\text{in}}\frac{1}{R}\sqrt{\frac{L}{C}}=-jQ\mathbf{V}_{\text{in}} \tag{25.12}$$

and we could write the gain at the resonant frequency as:

$$G = \frac{\mathbf{V_{out}}}{\mathbf{V_{in}}} == -jQ \tag{25.13}$$

This gives another way of thinking about the Q-factor (there are a lot of them): the Q-factor is the magnitude of the gain of this circuit at the resonant frequency. Note that's it's possible to have a Q factor of greater than one, which implies that this simple passive circuit is providing voltage gain at this frequency.

Another interesting result that can be derived from equation (25.13) is that the resonant frequency is the frequency at which the effect of the two poles in this circuit is a phase shift of 90 degrees.

The initial value theorem would then predict the output voltage just after a step input to be:

$$H(\infty) = \frac{\mathbf{V_{out}}}{\mathbf{V_{in}}} = \frac{1}{1 + j\infty RC + (j\infty)^2 LC} = \frac{1}{(j\infty)^2 LC} = 0 \tag{25.14}$$

which is entirely as expected, since you cannot suddenly change the voltage across a capacitor (that would take an infinite current), and the voltage across the capacitor just before the step input is zero.

The final value theorem predicts the output voltage a long time after a step input to be:

$$H(\infty) = \frac{\mathbf{V_{out}}}{\mathbf{V_{in}}} = \frac{1}{1 + j0RC + (j0)^2 LC} = \frac{1}{1} = 1 \tag{25.15}$$

and again this is intuitively reasonable, since a long time after the step input, the capacitor will have fully charged, and no further current will be flowing in the circuit, so the voltages across the resistor and inductor will be zero.

In terms of what the frequency response looks like, this depends largely on the Q factor of the two poles. Plotting for the same resonant frequency, but a range of values of Q-factor, gives the family of curves shown below:

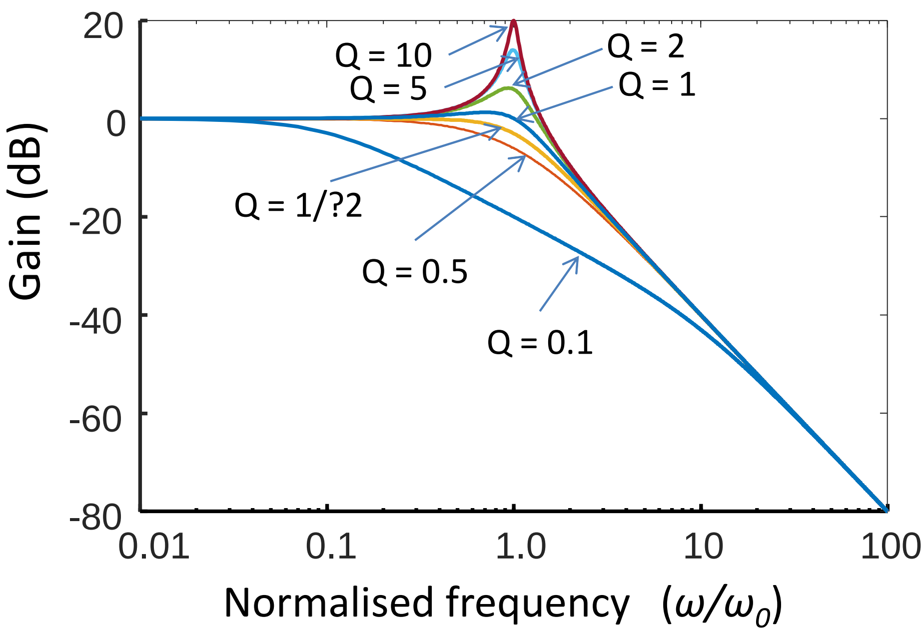

Figure 25.2 Response of second-order low-pass filter with various Q-factor values

It's worth thinking in a bit more detail about four special cases: very low Q-factors, a Q-factor of one-half, a Q-factor of 1/√2, and very high Q-factors.

25.3.1 The case of a very small Q-factor

A very small Q-factor corresponds to the case of the two poles both being real and a long way apart. This can perhaps most easily be seen by considering equation (25.4), which gives the relationship between Q-factor and the location of the two poles:

$$Q = -\frac{\sqrt{p_0 p_1}}{p_0 + p_1} = -\frac{\omega_0}{p_0 + p_1} \tag{25.16}$$

When one pole is a long way from the other one, (for example, the magnitude of p_0 is much greater than p_1), this can be approximated as:

$$Q = -\frac{\sqrt{p_0 p_1}}{p_0 + p_1} \approx Q = -\frac{\sqrt{p_0 p_1}}{p_0} = \sqrt{\frac{p_1}{p_0}} \qquad \therefore \frac{p_1}{p_0} \approx Q^2 \tag{25.17}$$

This also suggests that for very small Q values, the ratio of the poles is the square of the Q-factor. From this observation, we can derive:

$$\frac{p_1}{p_0} \approx Q^2 \qquad p_0 p_1 = \omega_0{}^2 \qquad \therefore p_0{}^2 = Q^2 p_1{}^2 \tag{25.18}$$

and from here it is simple to derive the approximate formula for the poles in this case of very low Q-factor (bearing in mind that both poles are negative and real in this case):

$$p_0 \approx -Q\omega_0 \qquad p_1 \approx -\frac{\omega_0}{Q} \tag{25.19}$$

For example, with a Q-factor of 0.1, and a resonant frequency of 10 kHz, this approximation suggests that the poles would have break frequencies at around 1 kHz and 100 kHz, in other words the lower break frequency is at about 1% of the frequency of the higher break frequency. This results in a frequency and phase response that looks like this:

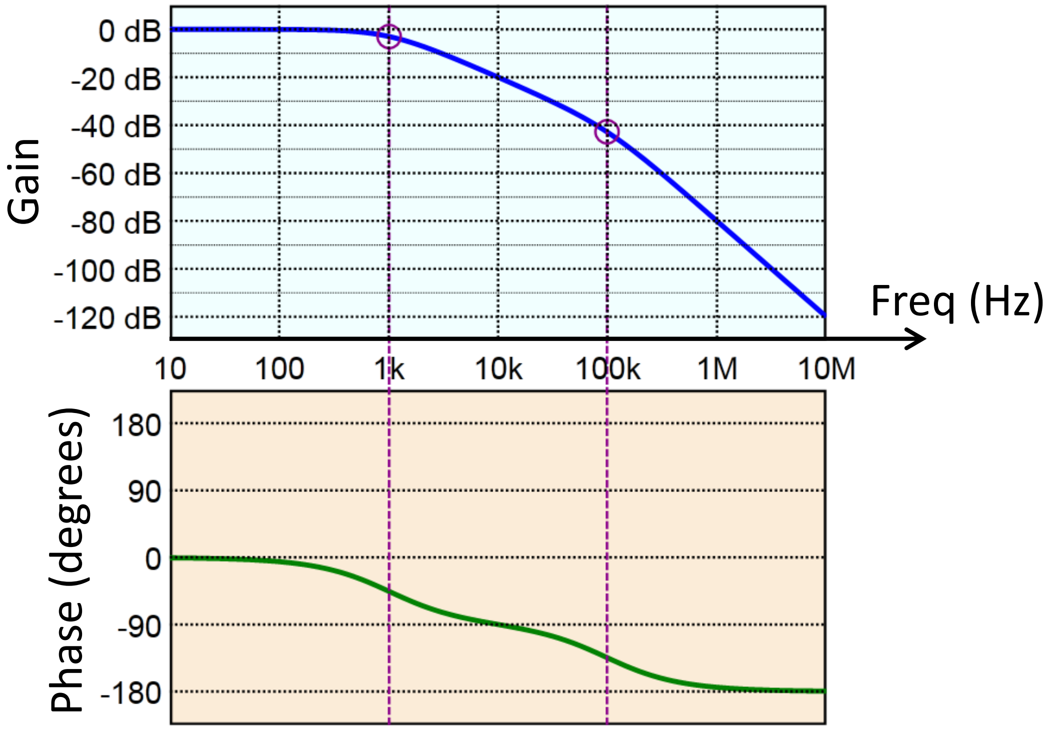

Figure 25.3 Frequency response for a Q-factor of 0.1

This is indeed borne out by the shape of the frequency response: from 10 Hz to 1 kHz it looks like the frequency response of a single-pole filter with a break frequency at 1 kHz; it's only when you approach the second pole's break frequency at 100 kHz that the phase starts reducing once again, and the gradient of the amplitude response increases from 20 dB/decade to 40 dB/decade.

Another way to think about this case is by considering the geometric model of the effect of poles and zeros in terms of the lines from the poles to a frequency point on the imaginary axis of an Argand diagram:

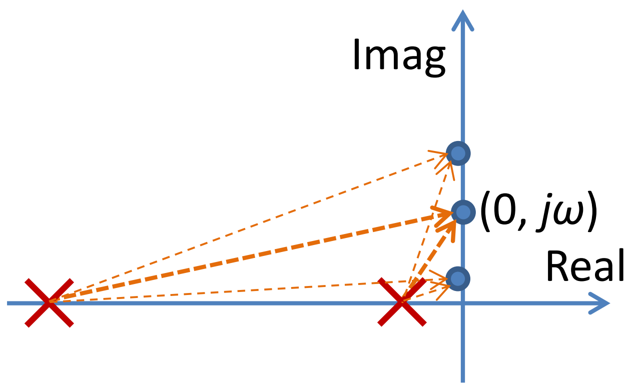

Figure 25.4 Poles on the Argand diagram for the case of Q = 0.1

The frequency point will be approximately the same distance and direction from the pole with the higher magnitude for most small frequencies, however the direction and distance from the lower magnitude pole will be changing considerably over this range of frequencies (by almost ninety degrees in the case of the direction). The change in gain and phase shift of the circuit at these frequencies will therefore be dominated by the effect of the lower magnitude pole.

Only when the frequency gets to approximately the same value as the higher magnitude pole's break frequency will this pole start to have an effect, and at this point the frequency is so far from the lower frequency pole that the gain of the circuit is already very small.

Yet another way to think about the effect of the poles is to construct the Bode approximation of the frequency response in this case. This suggests we can produce a good estimate of the response by starting from (in this case) a gain of 0 dB at low frequencies, then drawing a horizontal line to the first pole's break frequency, then bending the line down by 20 dB/decade, until it reaches the second pole's break frequency, at which point the slope increases by a further 20 dB/decade so it is now dropping at 40 dB/decade.

This effect can also be seen in Figure 25.3 above for the case where the Q-factor is 0.1.

25.3.1.1 The dominant pole approximation

Since the gain is already low by the time the frequency response reaches the higher frequency pole, a good approximation to the frequency response (and the step response) of these circuits is to consider only the pole with the lower break frequency, as this is the only one which has any significant effect on any frequencies which get through this filter.

This is known as the *dominant pole approximation*: the behaviour of this circuit is approximately the same as that of a first-order filter with only the lower-magnitude pole.

25.3.2 The case of a Q-factor of one-half

A Q-factor of one-half corresponds to the frequency response:

$$H(j\omega)=\frac{1}{1+(j\omega)\frac{1}{Q\omega_0}+\frac{(j\omega)^2}{\omega_0{}^2}}=\frac{1}{1+(j\omega)\frac{2}{\omega_0}+\frac{(j\omega)^2}{\omega_0{}^2}}=\frac{1}{\left(1+\frac{j\omega}{\omega_0}\right)^2} \tag{25.20}$$

This corresponds to the case where there are two poles both with break frequencies equal to the resonant frequency. Here the Bode approximation only has one turning point, but at this frequency the gradient of the line changes by 40 dB and the phase changes by 180 degrees since there are two poles contributing to the change.

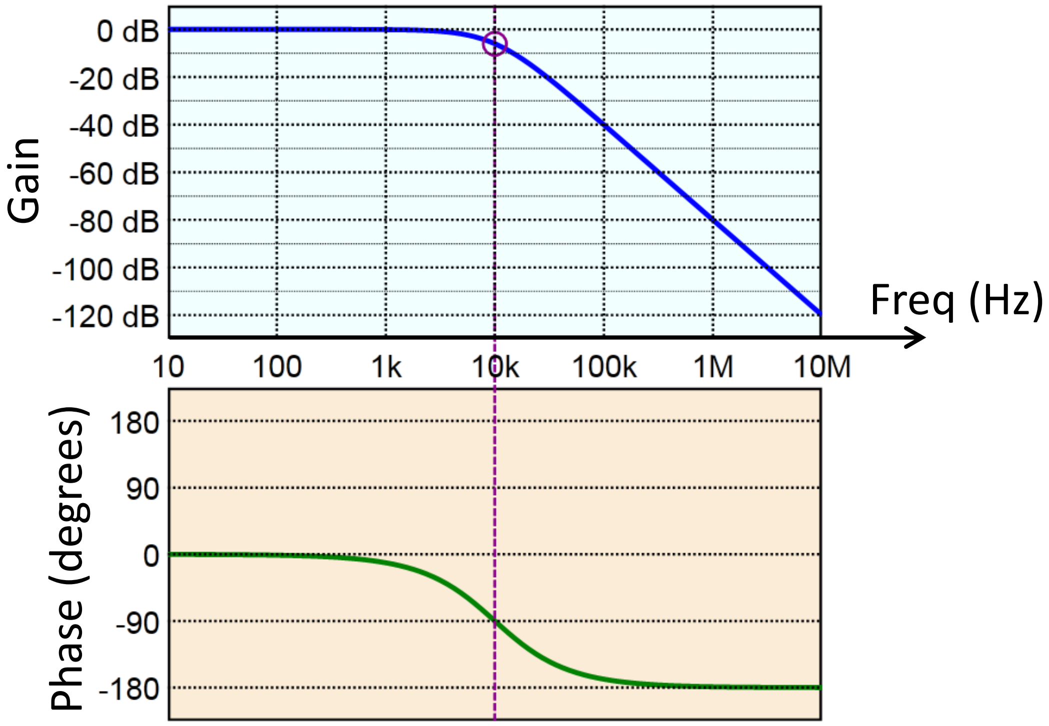

Figure 25.5 Frequency response of a second-order all-pole circuit with Q-factor of 0.5

Both the gain and the phase change by exactly twice as much as for a single-pole response (which is what should be expected).

25.3.3 The case of a Q-factor of one over the square root of two

This value is of interest, since it is the largest Q factor for which there is no peak in the frequency response (in other words the gain of the circuit is never greater than one).

After reading that last sentence you might be tempted to point out that I wrote in a previous section that the gain of the circuit at the resonant frequency is:

$$G = \frac{\mathbf{V}_{\mathbf{out}}}{\mathbf{V}_{\mathbf{in}}} == -jQ \tag{25.21}$$

so when the Q factor is one over root-two, surely the gain should be:

$$G = \frac{\mathbf{V}_{\mathbf{out}}}{\mathbf{V}_{\mathbf{in}}} == -j\frac{1}{\sqrt{2}} = -0.707j \tag{25.22}$$

which is quite clearly a long way from having a peak.

This isn't a contradiction. It is true that the gain of the circuit at the resonant frequency is *-jQ*, however the resonant frequency isn't necessarily the frequency at which the response has its maximum.

The actual maximum occurs when the denominator has a minimum value. This is perhaps easiest to determine from expressing the denominator in polar form:

$$1+(j\omega)\frac{1}{Q\omega_0}+\frac{(j\omega)^2}{\omega_0^{\;2}}=\left(1-\frac{\omega^2}{\omega_0^{\;2}}\right)+j\left(\frac{\omega}{Q\omega_0}\right)=\sqrt{\left(1-\frac{\omega^2}{\omega_0^{\;2}}\right)^2+\left(\frac{\omega}{Q\omega_0}\right)^2}\exp\left(j\tan^{-1}\left(\frac{\omega/Q\omega_0}{1-\frac{\omega^2}{\omega_0^{\;2}}}\right)\right) \quad (25.23)$$

and then looking for a turning point for the square of the magnitude[1] of the denominator:

$$\begin{aligned}
&\frac{d}{d\omega}\left(\left(1-\frac{\omega^2}{\omega_0^{\;2}}\right)^2+\left(\frac{\omega}{Q\omega_0}\right)^2\right)=0\\
&2\left(1-\frac{\omega^2}{\omega_0^{\;2}}\right)\left(\frac{-2\omega}{\omega_0^{\;2}}\right)+\frac{2\omega}{Q^2\omega_0^{\;2}}=0\\
&\frac{2\omega}{Q^2\omega_0^{\;2}}=\frac{4\omega}{\omega_0^{\;2}}\left(1-\frac{\omega^2}{\omega_0^{\;2}}\right)\\
&\frac{1}{Q^2}=2\left(1-\frac{\omega^2}{\omega_0^{\;2}}\right)\\
&\omega=\omega_0\sqrt{\left(1-\frac{1}{2Q^2}\right)}
\end{aligned} \quad (25.24)$$

For large values of Q the maximum is close to the resonant frequency, but for smaller values of Q, the maximum gain happens slightly below the resonant frequency.

[1] The choice to consider the square of the denominator makes the maths a little easier to do (you don't have to worry about the square-root) and it leads to the same result since if an expression has a turning point at a particular frequency, the square of the expression must have a turning point at that frequency as well.

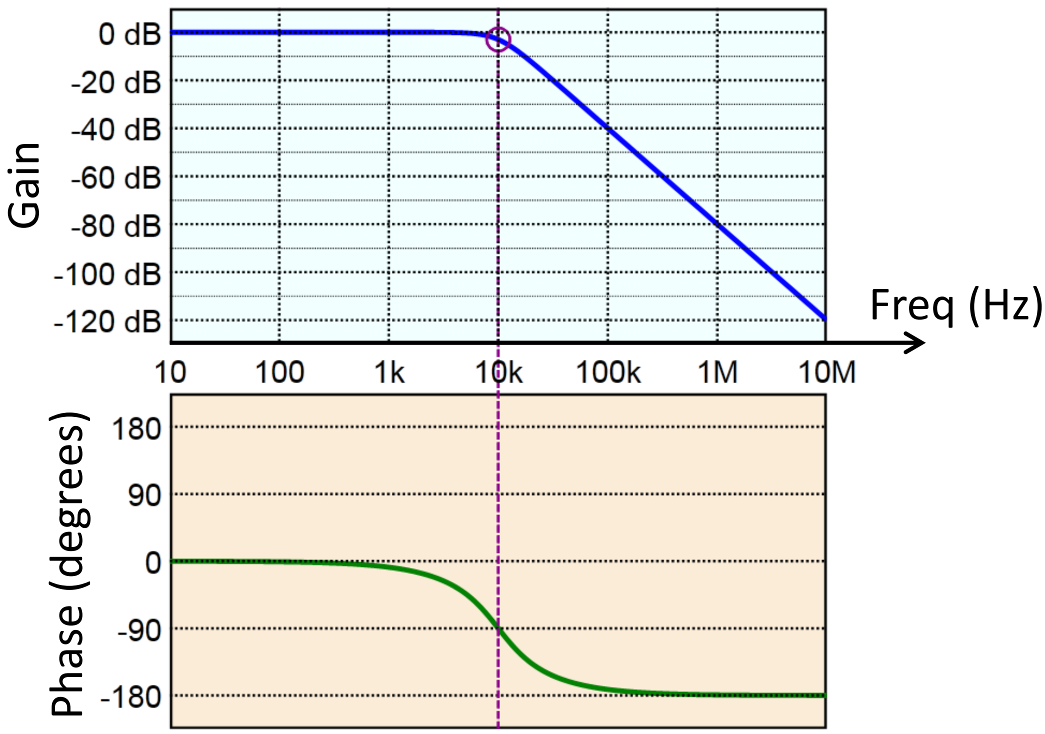

Figure 25.6 Frequency response for the case of Q = 1/√2

Looking at the expression for the frequency of the maximum:

$$\omega = \omega_0 \sqrt{\left(1 - \frac{1}{2Q^2}\right)} \tag{25.25}$$

you might notice that there are no real solutions when Q is less than 1/√2, and when Q is equal to 1/√2 the only solution is at zero. Only for values of Q of greater than 1/√2 does this equation have a non-zero, real solution.

Although the gain doesn't have a maximum turning-point here, the poles are complex, since from equation (25.3):

$$p_{0,1} = -\omega_0 \left(\frac{1 \pm \sqrt{1 - 4Q^2}}{2Q} \right) = -\omega_0 \left(\frac{1 \pm \sqrt{1-2}}{\sqrt{2}} \right) = -\omega_0 \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}} \right) = -\omega_0 \left(\frac{1 \pm j}{\sqrt{2}} \right) \tag{25.26}$$

so that when placed on the Argand diagram, they would appear at 45 degrees to the negative real axis, like this:

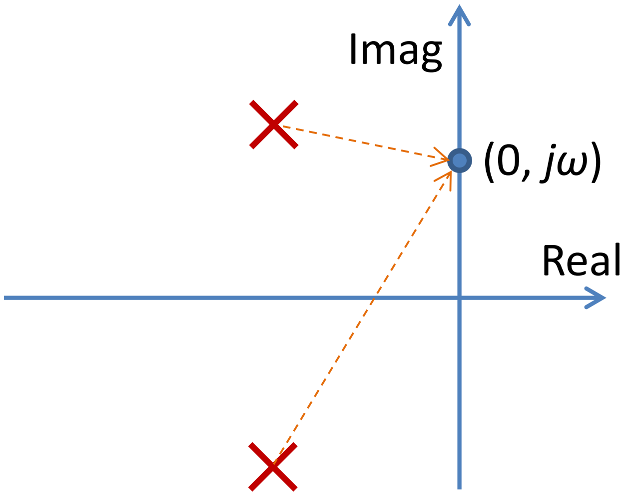

Figure 25.7 Poles for the case of Q = 1/√2 on an Argand diagram

25.3.4 The case of a large Q-factor

Large values of Q-factor are the most interesting case to consider, since here there really is peak in the frequency response: at some frequencies the output can be larger than the input.

This is perhaps surprising for a purely passive circuit, but it arises from the effect of the impedance of the capacitor and the inductor cancelling each other out, leaving the current limited only by the value of the resistor (which can be very small). However this large current will still produce a large voltage across the capacitor (and a large voltage in the opposite direction across the inductor).

The pole break frequencies in this case can again be derived using equation (25.3), however for the case of large Q, an alternate formula is commonly used. This can be derived by substituting cos(θ) = 1/2Q:

$$\begin{aligned}p_{0,1} &= -\omega_0\left(\frac{1\pm\sqrt{1-4Q^2}}{2Q}\right) = -\omega_0\left(\frac{1}{2Q}\pm\sqrt{\frac{1}{4Q^2}-1}\right)\\ &= -\omega_0\left(\frac{1}{2Q}\pm j\sqrt{1-\frac{1}{4Q^2}}\right) = -\omega_0\left(\cos\theta \pm j\sqrt{1-\cos^2\theta}\right)\\ &= -\omega_0\left(\cos\theta \pm j\sin\theta\right)\\ &= -\omega_0\exp\left(j\theta\right)\\ &= -\omega_0\exp\left(j\cos^{-1}\left(\frac{1}{2Q}\right)\right)\end{aligned} \qquad (25.27)$$

Note that this implies that the poles are complex, a distance ω_0 from the origin, with the angle to the real axis given by $\cos^{-1}(1/2Q)$, so the higher the value of Q, the closer the poles are to the imaginary axis. (In these cases it's perhaps no longer useful to think in terms of the break frequencies of the poles, and it's easier to stick to the resonant frequency and the Q-factor.)

Since we know the frequency at which the maximum gain occurs, the most important results left to derive in this case are how big the maximum gain is, and what the 3-dB bandwidth of the circuit is (in other words, over what range of frequencies does the circuit have a gain within 3-dB of its maximum gain). It turns out that there is a comparatively simple formula for the first of these results, but for the second we have to make some approximations to get a simple algebraic result.

The frequency response of a circuit with a Q-factor of 10 looks like this (note the fact that with high Q-values, the phase changes very suddenly):

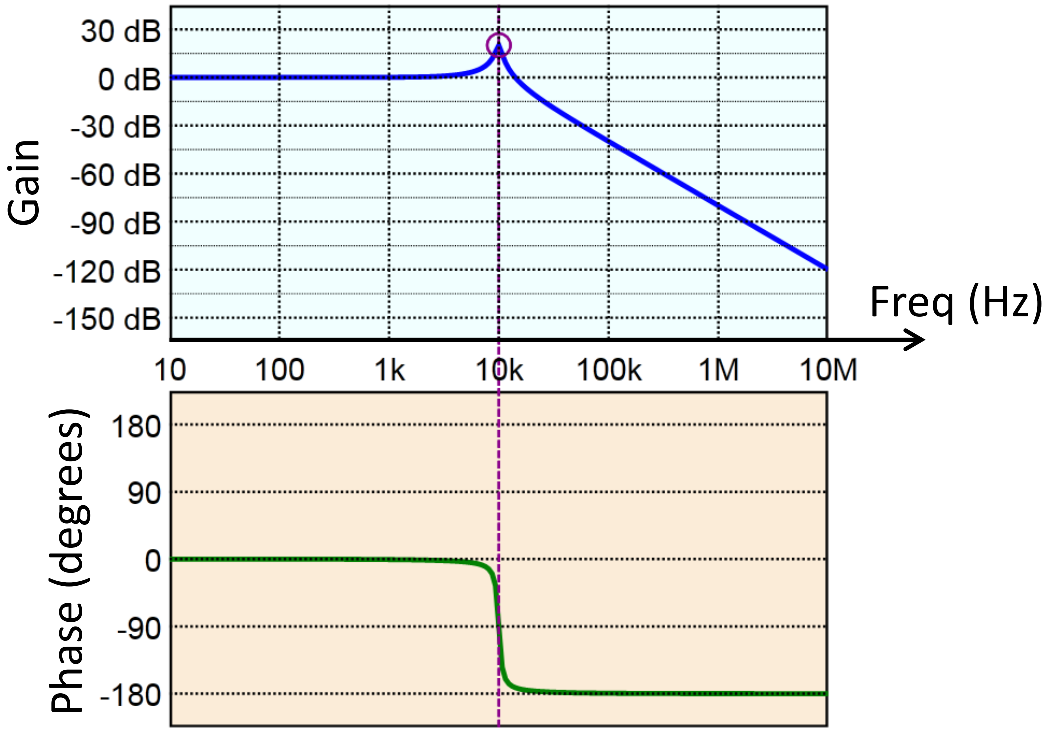

Figure 25.8 Frequency response of a second-order low-pass filter with Q = 10

The maximum gain can be determined by taking the formula for the frequency of the maximum gain (equation (25.24)) and substituting this back into the frequency response:

$$\begin{aligned}
H\left(\omega_0\sqrt{\left(1-\frac{1}{2Q^2}\right)}\right) &= \frac{1}{1+\left(j\omega_0\sqrt{\left(1-\frac{1}{2Q^2}\right)}\right)\frac{1}{Q\omega_0}+\frac{\left(j\omega_0\sqrt{\left(1-\frac{1}{2Q^2}\right)}\right)^2}{\omega_0{}^2}} \\
&= \frac{1}{1+\left(j\sqrt{\left(Q^2-\frac{1}{2}\right)}\right)\frac{1}{Q^2}+\left(j\sqrt{\left(1-\frac{1}{2Q^2}\right)}\right)^2} \\
&= \frac{1}{1+\left(j\sqrt{\left(4Q^2-2\right)}\right)\frac{1}{2Q^2}-\left(1-\frac{1}{2Q^2}\right)} \\
&= \frac{2Q^2}{1+j\sqrt{\left(4Q^2-2\right)}}
\end{aligned} \qquad (25.28)$$

Expressing this in polar terms (to get the magnitude) gives:

$$H\left(\omega_0\sqrt{\left(1-\frac{1}{Q^2}\right)}\right)=\frac{2Q^2}{\sqrt{1+4Q^2-2}}\exp\left(-j\sqrt{4Q^2-2}\right)=\frac{2Q^2}{\sqrt{4Q^2-1}}\exp\left(-j\sqrt{4Q^2-2}\right) \qquad (25.29)$$

giving the magnitude of the maximum gain as:

$$\left|H\left(\omega_{\text{max\_gain}}\right)\right|=\frac{2Q^2}{\sqrt{4Q^2-1}} \qquad (25.30)$$

Again, note that for large values of *Q* this is approximately equal to *Q*, but for smaller values of *Q* the maximum gain is slightly higher than Q (and occurs at a frequency slightly below the resonant frequency).

One last interesting quantity for this type of circuit is the 3-dB bandwidth of the resonant peak; in other words the range of frequencies for which the response is within a factor of √2 of the maximum gain. This can be determined by considering the frequencies for which the gain is √2 less than the maximum, which implies:

$$\left|H\left(\omega_{\text{max\_gain}}\right)\right|=\frac{\sqrt{2}Q^2}{\sqrt{4Q^2-1}} \qquad (25.31)$$

Using the result derived above for the square of the magnitude of the denominator, we can see that this happens when:

$$\frac{1}{\left(1-\frac{\omega^2}{\omega_0{}^2}\right)^2+\left(\frac{\omega}{Q\omega_0}\right)^2}=\frac{2Q^4}{4Q^2-1} \qquad (25.32)$$

This does not have a particularly neat solution, however the limiting case when *Q* is large does turn out to have a simple form[2]:

$$\Delta\omega_{3dB}\approx\frac{\omega_0}{Q} \qquad \text{for } Q\gg 1 \qquad (25.33)$$

25.4 A series RLC circuit with the resistor voltage as the output

If the output voltage is taken across the resistor instead of across the capacitor, a zero at zero is introduced into the frequency response.

[2] I'll leave the derivation to the interested reader; it's not required for this module, and it's rather tedious for those not so interested.

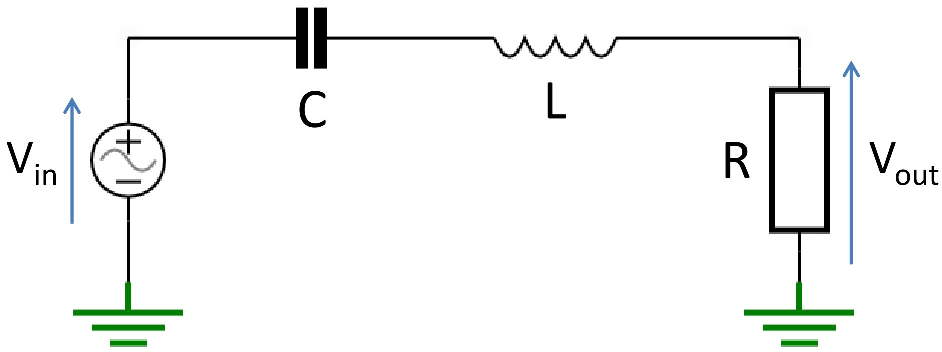

Figure 25.9 Circuit with a second-order response with a zero at zero Hz

We could predict that the output at very low frequencies will be very small (since almost no current will be flowing due to the very large impedance of the capacitor), and the output at very high frequencies will also be very small (this time almost no current will be flowing due to the very large impedance of the inductor). However, in-between these frequencies a substantial current will flow, particularly around the frequencies where the impedance of the capacitor and the inductor cancel out (the resonant frequency).

This suggests that there is now always a maximum gain in the response of this circuit, and a 3-dB bandwidth can always be determined. We just need to work out what the bandwidth and maximum gains are in terms of the component values.

To start, we can apply the potential divider equation in this circuit to produce an expression for the gain:

$$H(j\omega) = \frac{\mathbf{V}_{\mathbf{out}}}{\mathbf{V}_{\mathbf{in}}} = \frac{R}{R + j\omega L + 1/j\omega C} = \frac{j\omega RC}{1 + j\omega RC + (j\omega)^2 LC} \tag{25.34}$$

Note that the resonant frequency and the Q-factor of the poles are given by the same expressions as for the first circuit analysed:

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad Q = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{25.35}$$

however there is now a zero at zero, and a gain term which is equal to j/RC, so the frequency response could be expressed in terms of the resonant frequency and Q-factor of the poles as:

$$H(j\omega) = \frac{\dfrac{j\omega}{\omega_0 Q}}{1 + \dfrac{j\omega}{\omega_0 Q} + \dfrac{(j\omega)^2}{\omega_0^{\,2}}} \tag{25.36}$$

and once again, to determine the amplitude and phase characteristics of this response, it's useful to convert this into polar format, giving:

$$H(j\omega) = \frac{\dfrac{\omega}{\omega_0 Q}\exp\left(j\dfrac{\pi}{2}\right)}{\left(\left(1-\dfrac{\omega^2}{\omega_0{}^2}\right)+\dfrac{j\omega}{\omega_0 Q}\right)} = \frac{\dfrac{\omega}{\omega_0 Q}\exp\left(j\dfrac{\pi}{2}\right)}{\sqrt{\left(1-\dfrac{\omega^2}{\omega_0{}^2}\right)^2+\left(\dfrac{\omega}{\omega_0 Q}\right)^2}\exp\left(j\tan^{-1}\left(\dfrac{\dfrac{\omega}{\omega_0 Q}}{1-\dfrac{\omega^2}{\omega_0{}^2}}\right)\right)} \tag{25.37}$$

$$H(j\omega) = \frac{\dfrac{\omega}{\omega_0 Q}}{\sqrt{\left(1-\dfrac{\omega^2}{\omega_0{}^2}\right)^2+\left(\dfrac{\omega}{\omega_0 Q}\right)^2}}\exp\left(j\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{\omega_0}{Q\left(\omega_0{}^2-\omega^2\right)}\right)\right)\right) \tag{25.38}$$

which a bit of algebra and trigonometry can simplify to:

$$H(j\omega) = \frac{\omega_0\omega}{\sqrt{Q^2\left(\omega_0{}^2-\omega^2\right)^2+\omega_0{}^2\omega^2}}\exp\left(j\tan^{-1}\left(\frac{Q\left(\omega_0{}^2-\omega^2\right)}{\omega_0}\right)\right) \tag{25.39}$$

For two poles with break frequencies at 1 kHz and 100 kHz (a Q-factor of 0.1), the frequency response of this circuit looks like this:

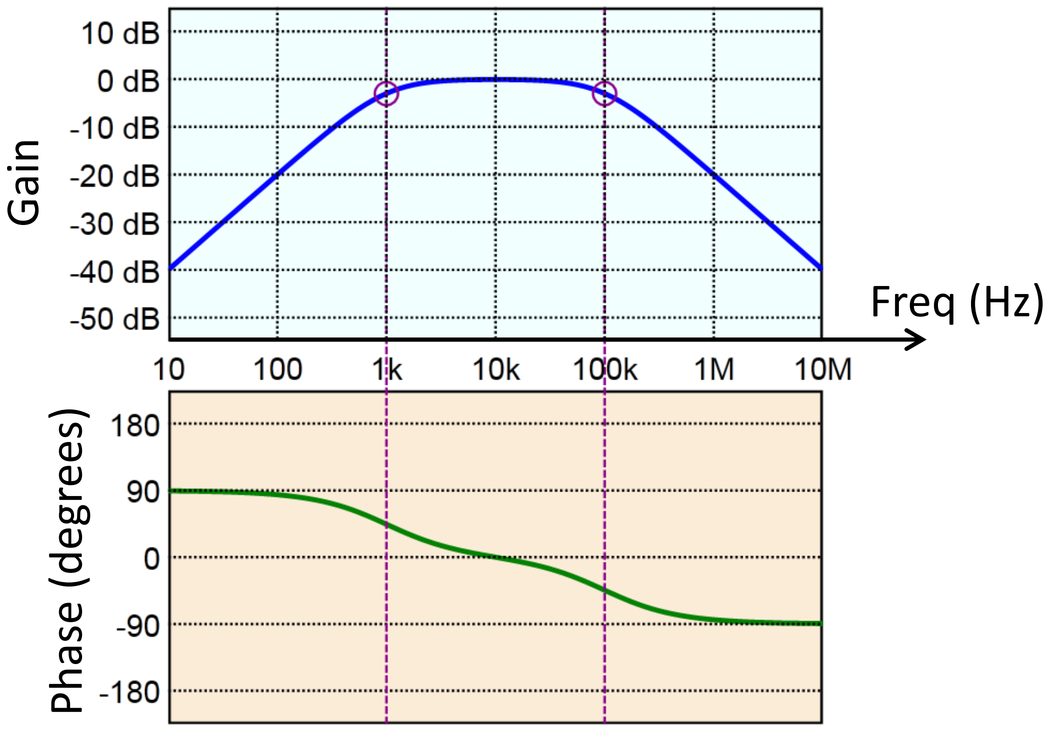

Figure 25.10 Frequency response of second-order bandpass filter with two real poles

whereas for a large Q-factor of 10, the frequency response looks like this:

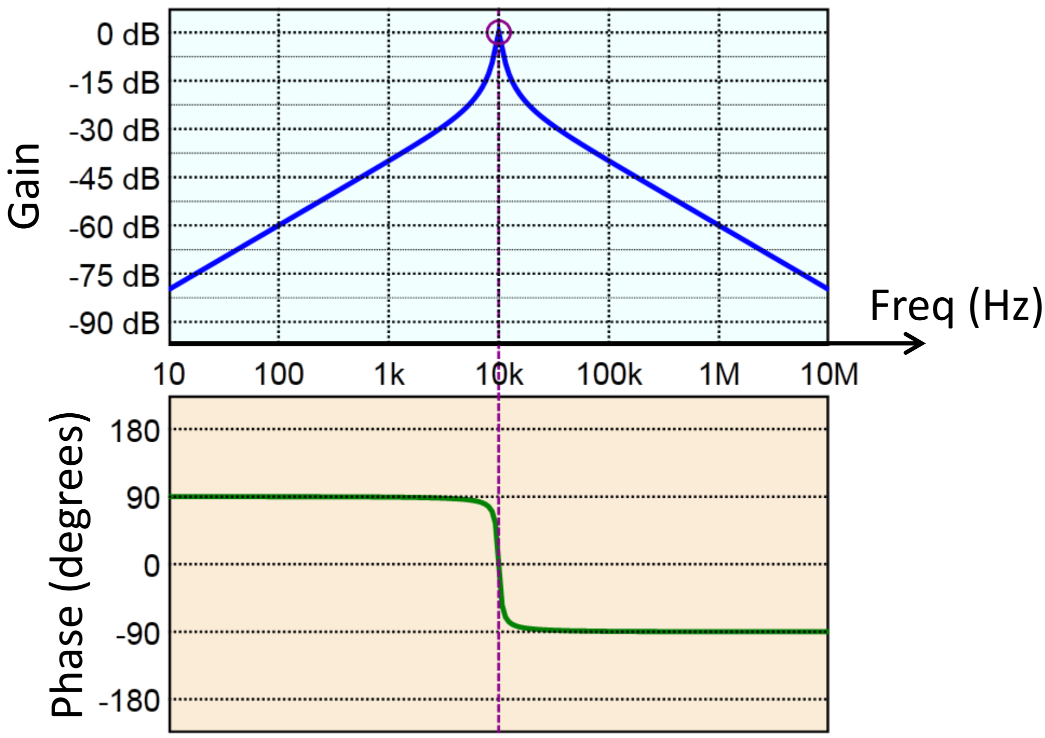

Figure 25.11 Frequency response of second-order bandpass filter with Q = 10

As expected, in both cases there is a maximum gain and a 3-dB bandwidth: a range of frequencies for which the gain is within 3 dB of the maximum gain.

For any bandpass filter, the most useful quantities to determine are the centre frequency (the frequency with the maximum gain), the gain at this centre frequency, and the 3-dB bandwidth (the range of frequencies for which the gain is within 3 dB of the maximum).

First, we'll find the maximum gain, and the frequency at which it occurs. At this point there are two ways to proceed: the straightforward brute-force "differentiating the magnitude of the frequency response and looking for a turning point" method, and the second method which requires a bit more thought: have a closer look at the expression for the square of the modulus of the frequency response (which gives a quantity proportional to the output power in the signal). From equation (25.39) we can determine this to be:

$$|H(j\omega)|^2 = \frac{\omega_0{}^2\omega^2}{Q^2\left(\omega_0{}^2-\omega^2\right)^2+\omega_0{}^2\omega^2} \tag{25.40}$$

If you think about it carefully, you might note that all three terms in the right-hand side fraction are squares, and that therefore they are all positive. You might also be able to see that the term $Q^2\left(\omega_0{}^2-\omega^2\right)^2$ is equal to zero when $\omega = \omega_0$, but at all other frequencies has a positive value. The equation above therefore has the form of:

$$\frac{A}{B+A} \tag{25.41}$$

where *A* and *B* are always zero or positive. The maximum possible value of this expression is one, and it occurs when $B = 0$, since for all other possible positive values of *B*, the expression evaluates to something less than one.

Therefore[3], the maximum possible gain of this circuit is one, and it occurs where $B = 0$, which happens when $\omega = \omega_0$.

The bandwidth is most commonly expressed in terms of the 3-dB bandwidth, in other words the range of frequencies for which the gain is within 3-dB of the maximum. Since the maximum in this case is one (or 0 dB), the 3-dB (or half-power) bandwidth occurs when the square of the frequency response is one-half. This occurs where:

$$\left|H(j\omega)\right|^2 = \frac{{\omega_0}^2\omega^2}{Q^2\left({\omega_0}^2-\omega^2\right)^2+{\omega_0}^2\omega^2} = \frac{1}{2} \tag{25.42}$$

and this is clearly going to occur where:

$$Q^2\left({\omega_0}^2-\omega^2\right)^2 = {\omega_0}^2\omega^2 \tag{25.43}$$

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[3] If you're struggling to follow that argument, then you could just do the differentiation of the square of the amplitude of the frequency response and look for a turning point.

This gives:

$$\frac{d\left|H(j\omega)\right|^2}{d\omega} = \frac{2{\omega_0}^2\omega\left(Q^2\left({\omega_0}^2-\omega^2\right)^2+{\omega_0}^2\omega^2\right)-{\omega_0}^2\omega^2\left(-4\omega Q^2\left({\omega_0}^2-\omega^2\right)+2{\omega_0}^2\omega\right)}{\left(Q^2\left({\omega_0}^2-\omega^2\right)^2+{\omega_0}^2\omega^2\right)^2}$$

Error! Main Document Only.(25.41)

which is zero when:

$$\begin{aligned}
2{\omega_0}^2\omega\left(Q^2\left({\omega_0}^2-\omega^2\right)^2+{\omega_0}^2\omega^2\right) &= {\omega_0}^2\omega^2\left(-4\omega Q^2\left({\omega_0}^2-\omega^2\right)+2{\omega_0}^2\omega\right)\\
2Q^2\left({\omega_0}^2-\omega^2\right)^2+2{\omega_0}^2\omega^2 &= -4\omega^2 Q^2\left({\omega_0}^2-\omega^2\right)+2{\omega_0}^2\omega^2\\
2Q^2\left({\omega_0}^2-\omega^2\right)^2 &= -4\omega^2 Q^2\left({\omega_0}^2-\omega^2\right)\\
\left({\omega_0}^2-\omega^2\right)^2 &= -2\omega^2\left({\omega_0}^2-\omega^2\right)\\
{\omega_0}^4-2\omega_0\omega^2+\omega^4 &= 2\omega^4-2\omega_0\omega^2\\
{\omega_0}^4 &= \omega^4
\end{aligned}$$

Error! Main Document Only.(25.41)

and since we know that both ω_0 and ω are real and positive, this means that the maximum gain (the centre of the bandpass response) must occur at $\omega = \omega_0$.

since as long as this is true, the denominator will be exactly twice the numerator. Taking the square root of both sides gives:

$$Q\left(\omega_0{}^2-\omega^2\right)=\pm\omega_0\omega \tag{25.44}$$

(note the introduction of the ± sign here, since we can't be sure at this stage which square root will lead to the solutions we're interested in).

This reveals a quadratic equation for the frequencies where the gain has dropped by 3-dB (ω):

$$Q\omega^2\pm\omega_0\omega-Q\omega_0{}^2=0 \tag{25.45}$$

Solving this quadratic gives:

$$\omega=\frac{\pm\omega_0\pm\sqrt{\omega_0{}^2+4Q^2\omega_0{}^2}}{2Q}=\omega_0\frac{\pm1\pm\sqrt{1+4Q^2}}{2Q} \tag{25.46}$$

There are a total of four different solutions for ω here, however only two of them are positive. We know that the frequencies we want are both positive, so just taking the two positive solutions gives:

$$\begin{aligned}\omega_1&=\omega_0\frac{+1+\sqrt{1+4Q^2}}{2Q}\\ \omega_2&=\omega_0\frac{-1+\sqrt{1+4Q^2}}{2Q}\end{aligned} \tag{25.47}$$

The bandwidth is the difference between these two frequencies, which is:

$$\begin{aligned}\Delta\omega=\omega_1-\omega_2&=\omega_0\left(\frac{+1+\sqrt{1+4Q^2}}{2Q}-\frac{-1+\sqrt{1+4Q^2}}{2Q}\right)\\ &=\omega_0\left(\frac{1}{2Q}-\frac{-1}{2Q}\right)=\omega_0\left(\frac{2}{2Q}\right)=\frac{\omega_0}{Q}\end{aligned} \tag{25.48}$$

This formula is another of the definitions of Q: for a two-pole bandpass filter with a single zero at zero, the Q-factor is the ratio of the centre frequency of the filter (the frequency of maximum gain) to the 3-dB bandwidth:

$$Q=\frac{\omega_0}{\Delta\omega}=\frac{\text{centre frequency}}{\text{3-dB bandwidth}} \tag{25.49}$$

Finally, there's another very simple and useful result that can be derived from equation (25.47): consider the product of the two 3-dB frequencies:

$$\omega_1\omega_2 = \left(\omega_0 \frac{+1+\sqrt{1+4Q^2}}{2Q}\right)\left(\frac{-1+\sqrt{1+4Q^2}}{2Q}\right)$$
$$= \omega_0^{\,2}\left(\frac{\sqrt{1+4Q^2}}{2Q}+\frac{1}{2Q}\right)\left(\frac{\sqrt{1+4Q^2}}{2Q}-\frac{1}{2Q}\right) \tag{25.50}$$

The right-hand side is the difference in two squares, so:

$$\omega_1\omega_2 = \omega_0^{\,2}\left(\frac{1+4Q^2}{4Q^2}-\frac{1}{4Q^2}\right) = \omega_0^{\,2}\left(\frac{1}{4Q^2}+1-\frac{1}{4Q^2}\right) = \omega_0^{\,2} \tag{25.51}$$

In other words, the resonant frequency is the geometric mean (the square root of the product) of the two 3-dB frequencies.

This is perhaps intuitively reasonable from looking at the plots of the amplitude response: they are symmetrical about the resonant frequency (the maximum gain) when plotted on with a logarithmic frequency axis. This implies that the ratio of the higher 3-dB frequency to the resonant frequency is equal to the ratio of the resonant frequency to the lower 3-dB frequency, and the result follows directly from that observation.

25.5 A variant on the bandpass circuit

There's another way to build a bandpass circuit which is common in practice. Consider the circuit below:

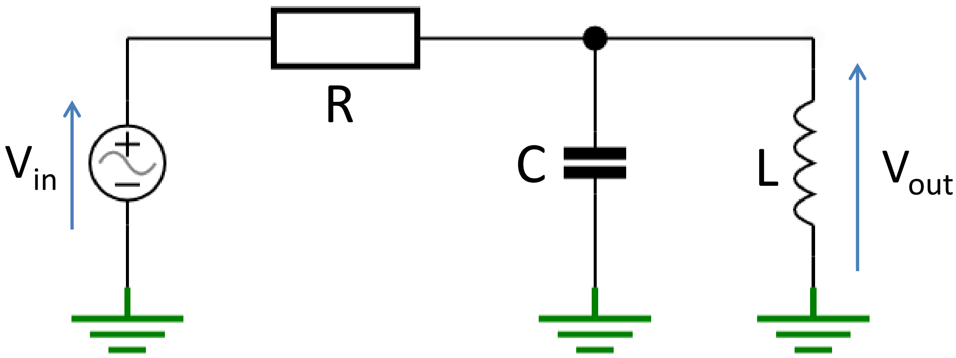

Figure 25.12 A variant of the bandpass filter circuit

Again using the potential divider equation, this circuit can be shown to have a frequency response given by:

$$H(j\omega) = \frac{\dfrac{j\omega L / j\omega C}{j\omega L + 1/j\omega C}}{R + \dfrac{j\omega L / j\omega C}{j\omega L + 1/j\omega C}} = \frac{j\omega \dfrac{L}{R}}{1 + j\omega \dfrac{L}{R} + (j\omega)^2 CL} \tag{25.52}$$

which you'll note now has:

$$Q = R\sqrt{\frac{C}{L}} \qquad \omega_0 = \frac{1}{\sqrt{LC}} \tag{25.53}$$

so the Q factor is the exact inverse from the previous circuit and this can make it easier in practice to produce circuits with very high Q factor values; however apart from this, the analysis is exactly the same as for the last case.

25.6 Summary: the most important things to know

- Second-order responses have two poles in their frequency response.
- They are usually specified in terms of their resonant frequency and Q-factor (rather than the location of the two poles).
- The resonant frequency is the frequency at which the high-frequency asymptote and lower-frequency asymptote meet.
- For low-pass or high-pass filters, the amplitude response at the resonant frequency is equal to the Q-factor.
- For the high-pass filter, the Q-factor is the ratio of the resonant frequency to the 3-dB bandwidth.
- For Q-factors above 0.5, the poles are complex; for Q-factors below 0.5 they are real. If the Q-factor is 0.5, the two poles are co-incident.