24 A Short Introduction to First-Order Responses

v1.7 – June 2021

Prerequisite knowledge required: Frequency Responses, Bode Plots

24.1 Introduction

This is the first of a the chapters in which we'll be taking a closer look at some of the most common frequency responses. The first-order response is particularly important, since it describes the frequency response of all of the op-amp amplifiers that we've been looking at, as well as some very simple but useful filter circuits.

To start with an example that we've seen before:

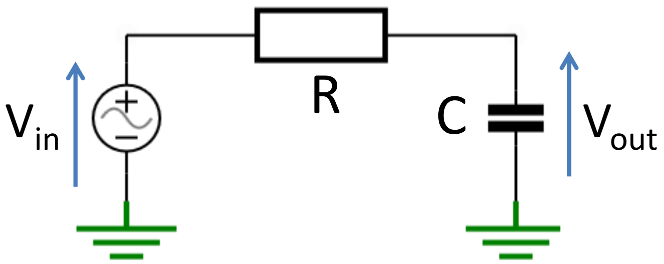

Figure 24.1 A simple RC low-pass filter

It has a frequency response (the ratio of the phasor representation of the output to the phasor representation of the input) given by:

$$H(j\omega) = \frac{\mathbf{V}_{\mathbf{out}}}{\mathbf{V}_{\mathbf{in}}} = \frac{1}{1 + j\omega RC} \tag{24.1}$$

This corresponds to a gain at low frequencies of one, and a single pole at $\omega = -1/RC$ rad/s which has a break frequency at $|\omega| = 1/RC$ rad/s.

This is a first-order response. Any frequency response for which the maximum power of $j\omega$ in the denominator polynomial is one (in other words it has one pole) is a first-order response.

Since the numerator can't have a larger order than the denominator (otherwise at infinite frequency the system would have infinite gain, and that's not possible) and there can't be a pole at zero (otherwise the system would have an infinite gain at DC), passive[1] circuits can only show four possible variations of first-order response:

- A single pole with a non-zero break frequency and no zeros

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[1] I have to add the word "passive" in here since it is (at least in theory) possible to produce circuits with infinite gain at higher frequencies or at zero Hz using active components. In the former case it's likely that some higher-frequency poles are being neglected in the analysis, in the latter case the circuits would sooner or later enter a non-linear region of operation (in which case all this theory falls apart).

However for passive circuits constructed from resistors, capacitors and inductors, these four types are the only possibilities.

- A single pole with a non-zero break frequency and a zero at zero
- A single pole with a non-zero break frequency and a zero with a non-zero break frequency lower than the pole's break frequency
- A single pole with a non-zero break frequency and a zero with a non-zero break frequency higher than the pole's break frequency

This short note will introduce each of these types of response, show how the breakpoints of the poles and zeros can be calculated, and how they affect the frequency and step responses of the circuits.

24.2 Step response and the initial and final value theorems

The step response of a circuit is the output when the input is a unit step (i.e. it starts at zero, and at time t = 0, suddenly increases to one). Unit steps[2] look like this:

Figure 24.2 The unit step function

It is possible to derive the step response from the frequency response, but this theory is beyond the syllabus of this module, so we'll just use differential equations and circuit analysis to get there.

While the whole theory will have to wait for a later chapter, there are a couple of results that derive from this theory that are useful to know at this stage. These are known as the *final value theorem* and the *initial value theorem*.

The final value theorem[3] states that the final value of the step response can be determined by evaluating the frequency response at $\omega = 0$.

The initial value theorem[4] states that the value of the step response just after $t = 0$ can be determined evaluating the frequency response at $\omega = \infty$.

These are useful to know, as they provide a simple way to check that the results from the differential equation are correct.

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[2] Known to mathematicians as the *Heaviside step function*.

[3] I should mention that this is a simplified form of the final value theorem which only applies to step responses. The full version of the final value theorem applies to any form of response, but that's beyond the syllabus for this module.

[4] Likewise, this is a simplified form of the initial value theorem, but it's all we need for now.

24.3 The four first-order responses

We can now go through the four possible first-order responses in turn, giving a circuit example and working out the frequency and step responses.

24.3.1 The first response: a single pole

This is the response we've already seen, it arises in circuits such as the one shown in Figure 24.1. The frequency response has the form:

$$H(j\omega) = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = \frac{1}{1 + j\omega/(-p)} \tag{24.2}$$

where p is the single pole in the response. It's often more useful to express this in terms of the break frequency of the pole ω_p, which is given by:

$$\omega_p = |p| = \frac{1}{RC} \tag{24.3}$$

(Note that this is the break frequency in radians/second[5], the break frequency in Hz is 1/(2πRC).)

Then to determine the amplitude and phase responses, we can express the frequency response in polar terms, which gives:

$$H(j\omega) = \frac{1}{1 + j\omega/\omega_p} = \frac{1}{\sqrt{1 + \omega^2/\omega_p^{\,2}}} \exp\left(-j\tan^{-1}\left(\frac{\omega}{\omega_p}\right)\right) \tag{24.4}$$

from which the amplitude response can be seen to be:

$$|H(j\omega)| = \frac{1}{\sqrt{1 + \omega^2/\omega_p^{\,2}}} \tag{24.5}$$

and the phase response:

$$\arg(H(j\omega)) = -\tan^{-1}\left(\frac{\omega}{\omega_p}\right) \tag{24.6}$$

Note that the frequency-dependent terms in these equations are ratios of the frequency to the pole's break frequency, so these could also be expressed in terms of the frequency in Hz, since $\omega / \omega_p = f / f_p$, where f_p is the break frequency of the pole in Hz:

$$|H(f)| = \frac{1}{\sqrt{1 + f^2/f_p^{\,2}}} \qquad \arg(H(f)) = -\tan^{-1}\left(\frac{f}{f_p}\right) \tag{24.7}$$

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[5] Another reminder: in these chapters I'll always use ω to refer to a frequency in radians per second, and f to refer to a frequency in hertz.

When plotted against the frequency on a logarithmic scale, these look like this:

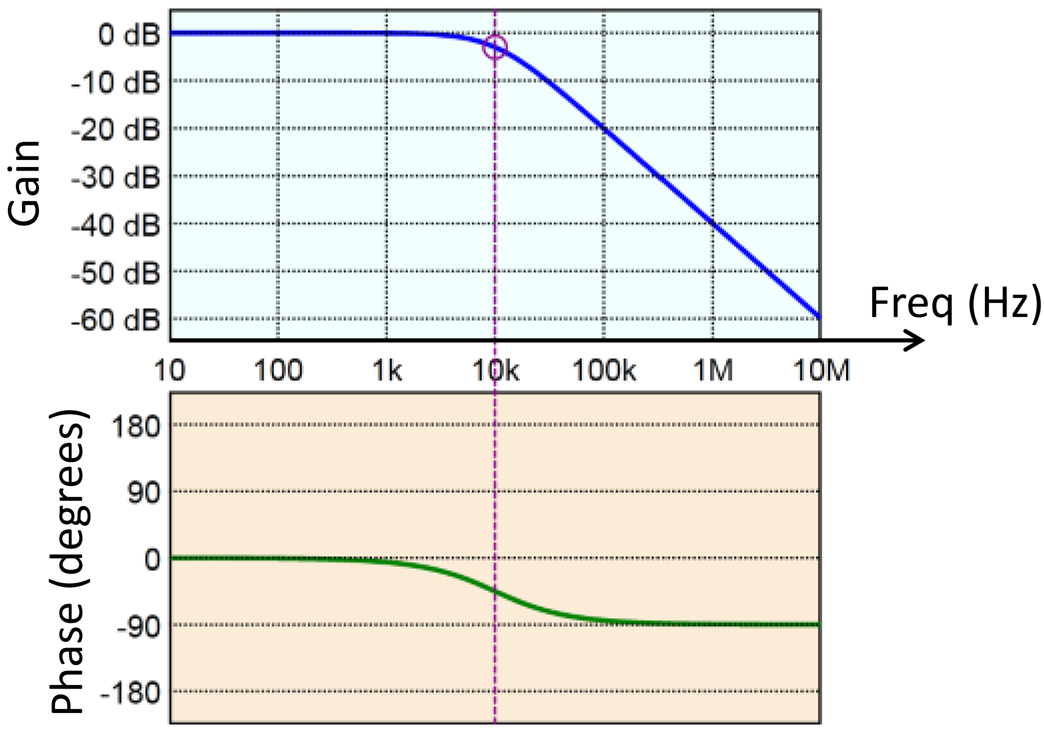

Figure 24.3 Amplitude and phase response of a single-pole response

All of this we've seen before; the new thing here is the step response. Since a step is not a cisoidal or sinusoidal signal, we can't use the theory of complex impedances to analyse the circuit any more. We have to go back to the original equations about the response of resistors and capacitors.

A nodal analysis of the circuit reveals that:

$$V_{in} - V_{out}(t) = I_R(t) \times R \tag{24.8}$$

$$I_C(t) = \frac{dQ(t)}{dt} = C\frac{dV_{out}(t)}{dt} \tag{24.9}$$

$$I_R(t) = I_C(t) \tag{24.10}$$

where I_R is the current through the resistor, I_C is the current through the capacitor and Q is the charge on the top plate of the capacitor, and all are functions of time.

Solving these for V_{out} reveals that:

$$V_{in} - V_{out}(t) = RC\frac{dV_{out}(t)}{dt} \tag{24.11}$$

Now since for the period of interest V_{in} is constant and equal to one (only positive times are interesting; at all negative times V_{out} and V_{in} are both zero):

$$\frac{dV_{out}(t)}{1-V_{out}(t)} = \frac{dt}{RC} \tag{24.12}$$

The initial value theorem here suggests that the initial value of V_{out} should be zero, since at infinite frequency:

$$\frac{1}{1+j\omega/p_b} = \frac{1}{1+j\infty/p_b} = \frac{1}{j\infty/p_b} = 0 \tag{24.13}$$

therefore using appropriate limits on the integration reveals that:

$$\begin{gathered}\int_0^{V_{out}(t)} \frac{dV_{out}(t)}{1-V_{out}(t)} = \int_0^t \frac{dt}{RC} \\ \left[-\ln\left(1-V_{out}(t)\right)\right]_0^{V_{out}(t)} = \left[\frac{t}{RC}\right]_0^t \\ -\ln\left(1-V_{out}(t)\right) = \frac{t}{RC} \\ V_{out}(t) = 1-\exp\left(\frac{-t}{RC}\right)\end{gathered} \tag{24.14}$$

This is an exponential rise, with a time-constant of *RC*, with an initial value of zero and a final value of one. It looks like this:

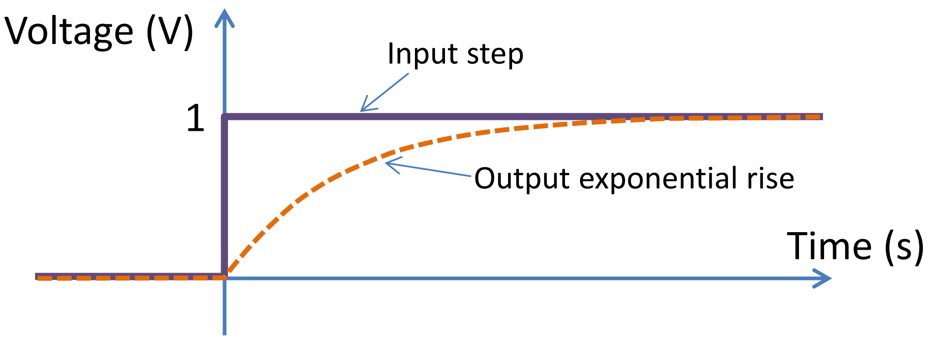

Figure 24.4 Step response of first-order system with no zeros

We can use the final value theorem to check this result. The frequency response is:

$$H(j\omega) = \frac{1}{1+j\omega/\omega_p} \tag{24.15}$$

When the frequency is zero, this gives:

$$\frac{1}{1+j\omega/\omega_p} = \frac{1}{1+j0/\omega_p} = \frac{1}{1} = 1 \tag{24.16}$$

which is indeed the final value of the response (as time approaches infinity).

24.3.2 The second response: a single pole with a zero at zero

This response is characteristic of an RC circuit, but where the output is taken as the voltage across the resistor, not across the capacitor, for example in this circuit:

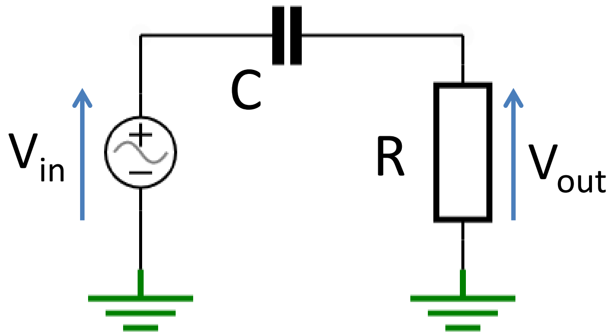

Figure 24.5 A simple RC high-pass filter circuit

Using the potential divider equation, we can readily work out the frequency response of this circuit, and find it to be:

$$H(j\omega) = \frac{\mathbf{V}_{\mathbf{out}}}{\mathbf{V}_{\mathbf{in}}} = \frac{Z_2}{Z_1 + Z_2} = \frac{R}{R + 1/j\omega C} = \frac{j\omega RC}{1 + j\omega RC} = \frac{j\omega/\omega_p}{1 + j\omega/\omega_p} \tag{24.17}$$

where again ω_p is the break frequency of the single pole in the response (at $1/RC$). Again, to determine the amplitude and phase responses, it's most convenient to express this function in polar terms, which gives:

$$\begin{aligned} H(j\omega) &= \frac{j\omega/\omega_p}{1 + j\omega/\omega_p} = \frac{\left(\dfrac{\omega}{\omega_p}\right)\exp\left(j\dfrac{\pi}{2}\right)}{\sqrt{1 + \omega^2/\omega_p^{\,2}}\exp\left(j\tan^{-1}\left(\dfrac{\omega}{\omega_p}\right)\right)} \\ &= \frac{\omega}{\omega_p\sqrt{1 + \omega^2/\omega_p^{\,2}}}\exp\left(j\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{\omega}{\omega_p}\right)\right)\right) \end{aligned} \tag{24.18}$$

from which the amplitude response can be seen to be:

$$\left|H(j\omega)\right| = \frac{\omega/\omega_p}{\sqrt{1 + \omega^2/\omega_p^{\,2}}} \tag{24.19}$$

and the phase response:

$$\arg\left(H(j\omega)\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega}{\omega_p}\right) \tag{24.20}$$

And once again, since $\omega / \omega_p = f / f_p$, where f_p is the break frequency of the pole in Hz, these can be expressed in terms of the frequency in Hz:

$$|H(f)| = \frac{f / f_p}{\sqrt{1 + f^2 / f_p^2}} \qquad \arg(H(f)) = \frac{\pi}{2} - \tan^{-1}\left(\frac{f}{f_p}\right) \tag{24.21}$$

When plotted against the frequency on a logarithmic scale, these look like this:

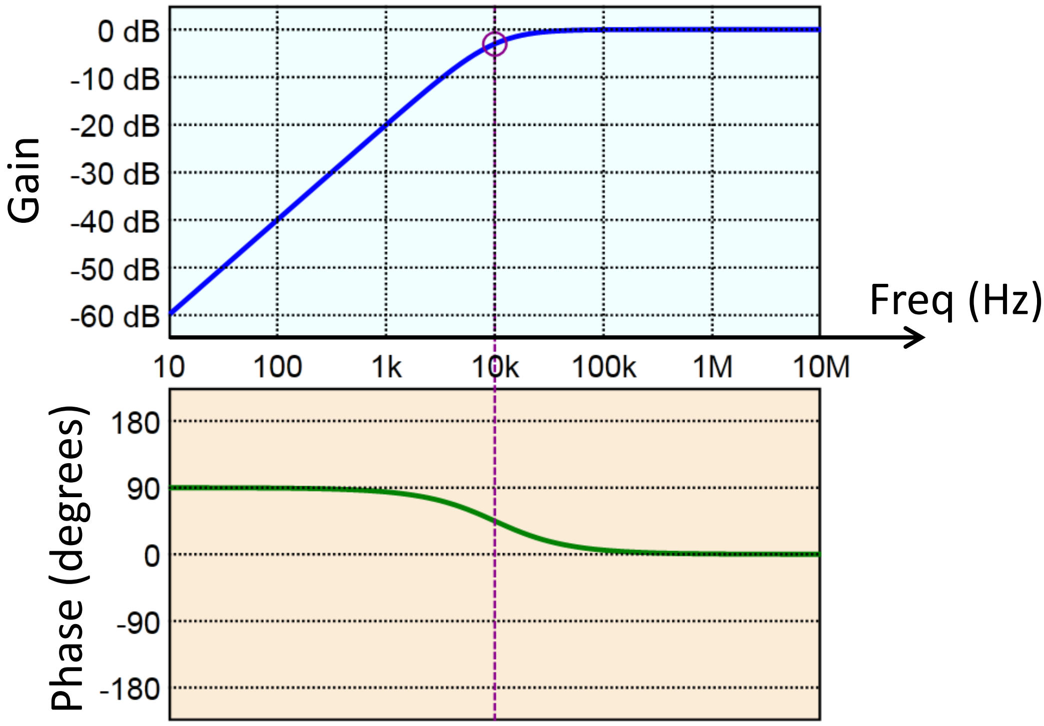

Figure 24.6 Amplitude and phase response of a single-pole response with a zero at zero Hz

This time the phase difference between the output and input starts at ninety degrees, and moves back to zero as the gain approaches 0 dB.

Now the step response: this time a nodal analysis of the circuit reveals that:

$$V_{out}(t) = I_R(t) \times R \tag{24.22}$$

$$I_C(t) = \frac{dQ(t)}{dt} = C\frac{d(V_{in} - V_{out}(t))}{dt} \tag{24.23}$$

$$I_R(t) = I_C(t) \tag{24.24}$$

where I_R is the current through the resistor, I_C is the current through the capacitor and Q is the charge on the top plate of the capacitor, and all these are functions of time.

Solving these for V_{out} reveals that:

$$V_{out}(t) = RC\frac{d(V_{in} - V_{out}(t))}{dt} = RC\frac{dV_{in}}{dt} - RC\frac{dV_{out}(t)}{dt} \tag{24.25}$$

Since for the period of interest V_{in} is constant, (again only positive times are interesting; at all negative times V_{out} and V_{in} are both zero):

$$V_{out}(t) = -RC\frac{dV_{out}(t)}{dt} \tag{24.26}$$

The initial value theorem here suggests that the initial value should be one, since at infinite frequency:

$$\frac{j\omega/\omega_p}{1+j\omega/\omega_p} = \frac{j\infty/\omega_p}{1+j\infty/\omega_p} = \frac{j\infty/\omega_p}{j\infty/\omega_p} = 1 \tag{24.27}$$

and using appropriate limits on the integration reveals that:

$$\begin{aligned}
\int_1^{V_{out}(t)} \frac{dV_{out}(t)}{V_{out}(t)} &= -\int_0^t \frac{dt}{RC} \\
\Big[\ln\big(V_{out}(t)\big)\Big]_1^{V_{out}(t)} &= -\left[\frac{t}{RC}\right]_0^t \\
\ln\big(V_{out}(t)\big) &= \frac{-t}{RC} \\
V_{out}(t) &= \exp\left(\frac{-t}{RC}\right)
\end{aligned} \tag{24.28}$$

This is an exponential decay, with a time-constant of *RC*, with an initial value of one and a final value of zero. It looks like this:

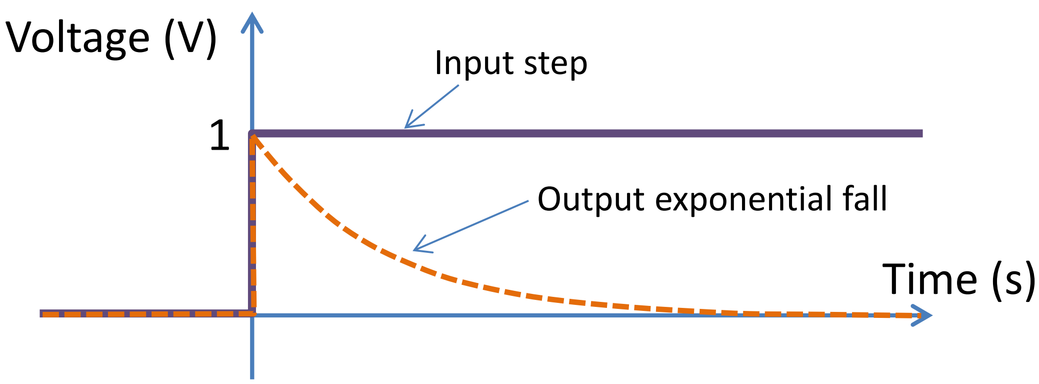

Figure 24.7 Step response of first-order system with no zeros

We can use the final value theorem to check this result (we've already used the initial value theorem to derive the limits on the integration). The frequency response is:

$$H(j\omega) = \frac{j\omega/\omega_p}{1+j\omega/\omega_p} \tag{24.29}$$

so when the frequency is zero, this gives:

$$\frac{j\omega/\omega_p}{1+j\omega/\omega_p}=\frac{0/\omega_p}{1+j0/\omega_p}=\frac{0}{1}=0 \tag{24.30}$$

which is indeed the final value of the response (as time approaches infinity).

24.3.3 The third response: a pole and zero with the pole's break frequency at a higher frequency than the zero's break frequency

To get a pole and a zero at different, non-zero frequencies, at least three components are required. In the example below, the circuit uses two resistors and one capacitor.

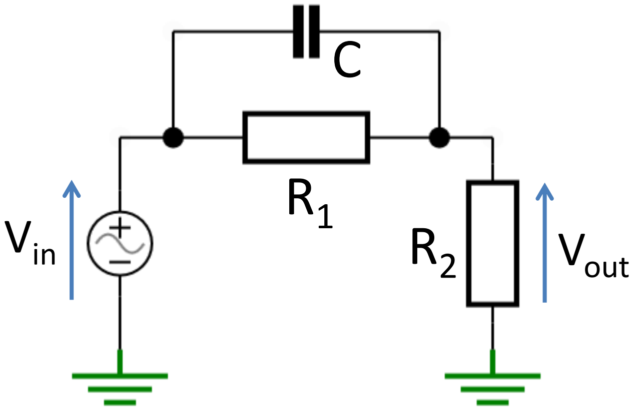

Figure 24.8 A circuit with a non-zero zero and a higher frequency pole

Once again, using the potential divider equation, we can work out the frequency response of this circuit. It's a bit more complicated this time, but the principle of using the potential divider equation still applies:

$$\begin{aligned} H(j\omega)=\frac{V_{out}}{V_{in}}=\frac{Z_2}{Z_1+Z_2}&=\frac{R_2}{R_2+\dfrac{R_1/j\omega C}{R_1+1/j\omega C}} \\ &=\frac{R_2(R_1+1/j\omega C)}{R_1R_2+R_1/j\omega C+R_2/j\omega C} \\ &=\frac{R_2(1+j\omega R_1C)}{R_1+R_2+j\omega R_1R_2C} \\ &=\frac{R_2}{R_1+R_2}\frac{(1+j\omega R_1C)}{\left(1+j\omega\dfrac{R_1R_2C}{R_1+R_2}\right)} \end{aligned} \tag{24.31}$$

This suggests there is one pole at p with a break frequency at:

$$\omega_p=|p|=\frac{R_1+R_2}{R_1R_2C} \tag{24.32}$$

and a zero at z with a break frequency at:

$$\omega_z = |z| = \frac{1}{R_1 C} \tag{24.33}$$

with a low-frequency gain of:

$$G = \frac{R_2}{R_1 + R_2} \tag{24.34}$$

It's worth noting that this low-frequency response is exactly what we'd have got for the gain of the circuit if the capacitor wasn't there; and this is entirely expected since at low-frequencies the capacitor has a very large impedance and can be neglected: the circuit really does behave like a simple two-resistor potential divider.

It's also no co-incidence that:

$$G = \frac{z_b}{p_b} \tag{24.35}$$

since this is required for the circuit to have a gain of one at very high frequencies (which it clearly would, since at very high frequencies the capacitor has a very small impedance).

In terms of the pole and zero break frequencies, the frequency response can therefore be written:

$$H(j\omega) = \frac{\omega_z}{\omega_p} \frac{(1 + j\omega/\omega_z)}{(1 + j\omega/\omega_p)} \tag{24.36}$$

Again, to determine the amplitude and phase responses, it's most convenient to express this function in polar terms, which gives:

$$\begin{aligned} H(j\omega) &= \frac{\omega_z}{\omega_p} \frac{(1 + j\omega/\omega_z)}{(1 + j\omega/\omega_p)} = \frac{\omega_z}{\omega_p} \frac{\sqrt{1 + \omega^2/\omega_z^2}\, \exp\left(j \tan^{-1}\left(\frac{\omega}{\omega_z} \right) \right)}{\sqrt{1 + \omega^2/\omega_p^2}\, \exp\left(j \tan^{-1}\left(\frac{\omega}{\omega_p} \right) \right)} \\ &= \frac{\omega_z}{\omega_p} \frac{\sqrt{1 + \omega^2/\omega_z^2}}{\sqrt{1 + \omega^2/\omega_p^2}} \exp\left(j\left(\tan^{-1}\left(\frac{\omega}{\omega_z} \right) - \tan^{-1}\left(\frac{\omega}{\omega_p} \right) \right) \right) \end{aligned} \tag{24.37}$$

from which the amplitude response can be seen to be:

$$|H(j\omega)| = \frac{\omega_z}{\omega_p} \sqrt{\frac{1 + \omega^2/\omega_z^2}{1 + \omega^2/\omega_p^2}} = \sqrt{\frac{\omega_z^2 + \omega^2}{\omega_p^2 + \omega^2}} \tag{24.38}$$

which clearly starts at ω_z/ω_p at low frequencies, and increases to one at high frequencies.

The phase response is given by:

$$\arg\left(H(j\omega)\right) = \tan^{-1}\left(\frac{\omega}{\omega_z}\right) - \tan^{-1}\left(\frac{\omega}{\omega_p}\right) \tag{24.39}$$

When plotted against the frequency on a logarithmic scale, these look like this (for the case where the zero's break frequency is 1% of the pole's break frequency so $\omega_z/\omega_p = 0.01$):

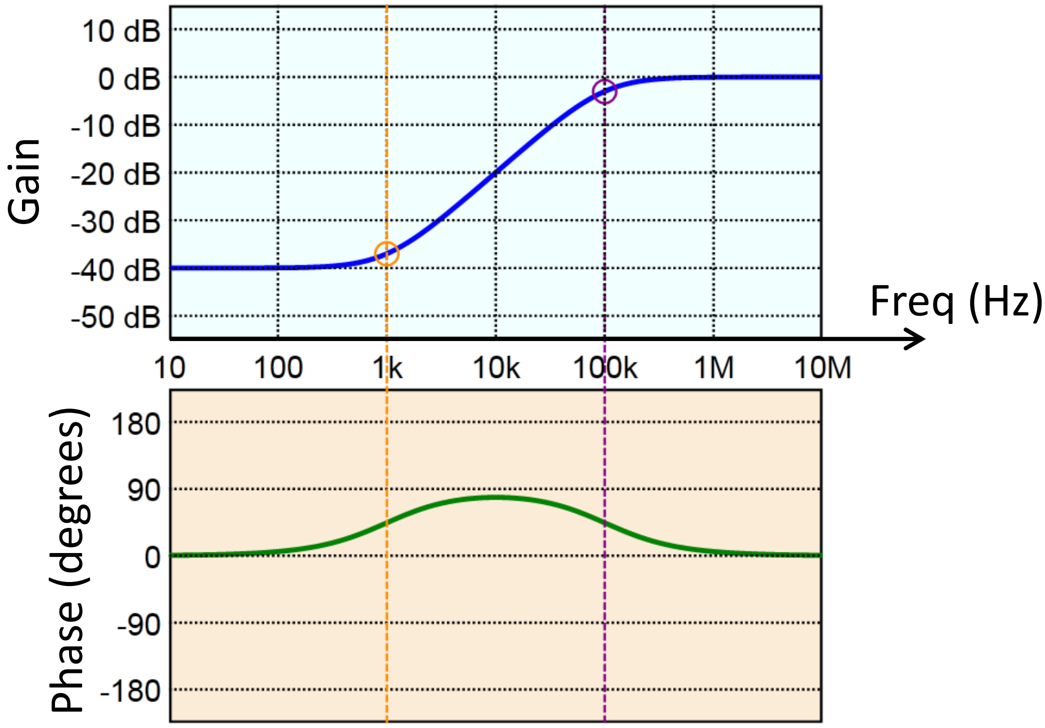

Figure 24.9 Amplitude and phase response of a non-zero zero and a higher frequency pole

At low frequencies the phase is zero, at high frequencies the phase is zero as well. It is only in-between the zero and pole break frequencies, where the two terms in equation (24.39) do not cancel out, that the phase response of this circuit does anything interesting.

On to the step response, which is again a little more complicated for this circuit. A nodal analysis reveals that:

$$V_{in} - V_{out}(t) = I_{R1}(t) \times R_1 \tag{24.40}$$

$$V_{out}(t) = I_{R2}(t) \times R_2 \tag{24.41}$$

$$I_C(t) = \frac{dQ(t)}{dt} = C\frac{d\left(V_{in} - V_{out}(t)\right)}{dt} \tag{24.42}$$

$$I_{R2}(t) = I_C(t) + I_{R1}(t) \tag{24.43}$$

where I_{R1} is the current through R_1, I_{R2} is the current through R_2, I_C is the current through the capacitor and Q is the charge on the top plate of the capacitor.

Solving these for V_{out} reveals that:

$$V_{out}(t)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)=\frac{1}{R_1}-C\frac{dV_{out}(t)}{dt} \tag{24.44}$$

At this point we could proceed with a change of variable, but it's usually easier to make an educated guess about the form of the solution, and see if it works. Here, I'll assume the solution has the form:

$$V_{out}(t)=A+B\exp\left(-\frac{t}{T}\right) \tag{24.45}$$

and substitute this into equation (24.44), then look for values of A, B, and T which work. The substitution gives:

$$\left(A+B\exp\left(-\frac{t}{T}\right)\right)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)=\frac{1}{R_1}+\frac{CB}{T}\exp\left(-\frac{t}{T}\right) \tag{24.46}$$

and expanding terms gives:

$$A\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+B\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\exp\left(-\frac{t}{T}\right)=\frac{1}{R_1}+\frac{CB}{T}\exp\left(-\frac{t}{T}\right) \tag{24.47}$$

For this to be true at any time, the constant values have to be equal, so:

$$\begin{gathered}A\left(\frac{1}{R_1}+\frac{1}{R_2}\right)=\frac{1}{R_1}\\ A=\frac{R_2}{(R_1+R_2)}\end{gathered} \tag{24.48}$$

and the coefficients of exp(-t/T) have to be equal, so:

$$B\left(\frac{1}{R_1}+\frac{1}{R_2}\right)=\frac{CB}{T} \tag{24.49}$$

From this we can deduce that the time-constant T must be:

$$T=\frac{R_1R_2C}{R_1+R_2} \tag{24.50}$$

but we can't deduce a unique value for B: anything would do. We'll have to look for a boundary condition to help with this one. A suitable boundary condition would be to note that according to the initial value theorem, the value of V_{out} just after t = 0 will be one, since:

$$H(\infty)=\frac{z_b}{p_b}\frac{(1+j\infty/z_b)}{(1+j\infty/p_b)}=\frac{z_b}{p_b}\frac{(j\infty/z_b)}{(j\infty/p_b)}=\frac{j\infty}{j\infty}=1 \tag{24.51}$$

and since:

$$V_{out}\left(t=0_+\right)=A+B\exp\left(-\frac{0}{T}\right)=A+B=1 \tag{24.52}$$

we can easily deduce that $B = 1 - A$.

This makes the final time-domain solution:

$$V_{out}\left(t\right)=\frac{R_2}{\left(R_1+R_2\right)}+\frac{R_1}{\left(R_1+R_2\right)}\exp\left(-\frac{t\left(R_1+R_2\right)}{R_1R_2C}\right) \tag{24.53}$$

This is an exponential fall, with a time-constant of $R_1R_2C/(R_1+R_2)$, with an initial value of one and a final value of $R_2/(R_1+R_2)$. It looks like this:

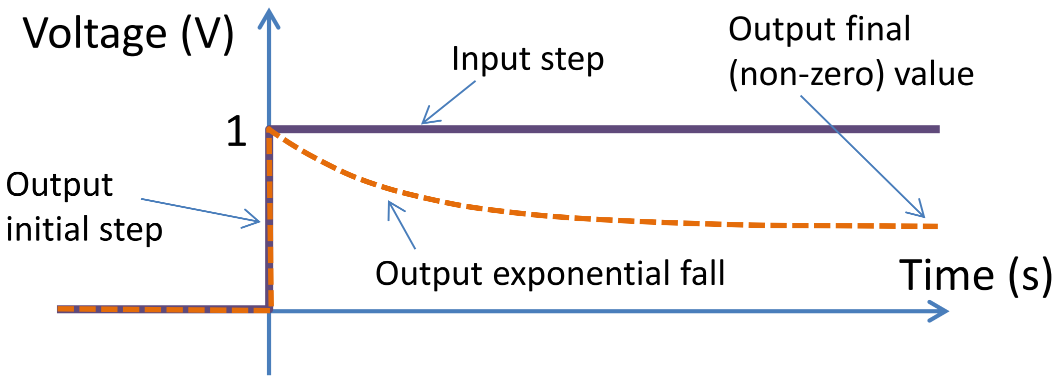

Figure 24.10 Step response of first-order system with one pole and a zero with a higher break frequency

We can use the final value theorem to check this result. The frequency response is:

$$H\left(j\omega\right)=\frac{\omega_z}{\omega_p}\frac{\left(1+j\omega/\omega_z\right)}{\left(1+j\omega/\omega_p\right)} \tag{24.54}$$

and when the frequency is zero, this gives:

$$H\left(0\right)=\frac{\omega_z}{\omega_p}\frac{\left(1+j0/\omega_z\right)}{\left(1+j0/\omega_p\right)}=\frac{\omega_z}{\omega_p} \tag{24.55}$$

and since:

$$\omega_z=\frac{1}{R_1C} \qquad \omega_p=\frac{R_1+R_2}{R_1R_2C} \tag{24.56}$$

we get:

$$\frac{\omega_z}{\omega_p}=\frac{R_2}{R_1+R_2} \tag{24.57}$$

which is the expected final value as t becomes infinite, since:

$$V_{out} = \frac{R_2}{(R_1 + R_2)} + \frac{R_1}{(R_1 + R_2)} \exp(-\infty) \to \frac{R_2}{(R_1 + R_2)} \tag{24.58}$$

24.3.4 The fourth response: a pole and zero with the zero's break frequency at a higher frequency than the pole's break frequency

Again, with a pole and a zero with different, non-zero break frequencies, at least three components are required. In the example below, the circuit uses two resistors and one capacitor.

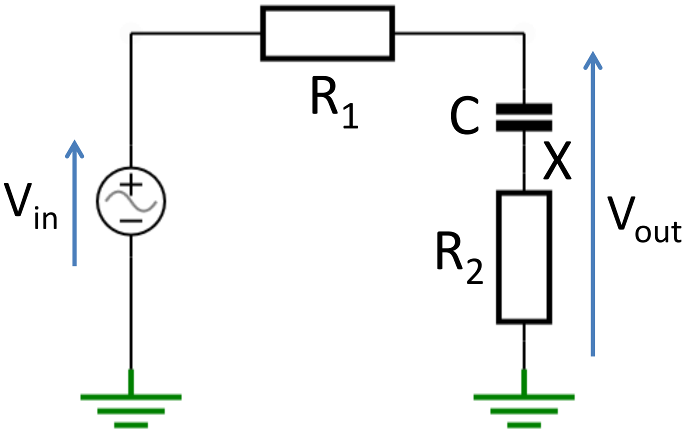

Figure 24.11 A circuit with a non-zero zero and a lower frequency pole

Once again, using the potential divider equation, we can work out the frequency response of this circuit. It's a bit simpler than last time:

$$\begin{aligned} H(j\omega) &= \frac{V_{out}}{V_{in}} = \frac{Z_2}{Z_1 + Z_2} = \frac{1/j\omega C + R_2}{R_2 + 1/j\omega C + R_2} \\ &= \frac{(1 + j\omega R_2 C)}{(1 + j\omega (R_1 + R_2) C)} \end{aligned} \tag{24.59}$$

This suggests there is one pole with a break frequency of:

$$\omega_p = \frac{1}{(R_1 + R_2)C} \tag{24.60}$$

and a zero with a break frequency of:

$$\omega_z = \frac{1}{R_2 C} \tag{24.61}$$

and a low-frequency gain of one. (The low-frequency gain of one is expected, since at low frequencies the capacitor has a very large impedance.)

In terms of the pole and zero break frequencies, the frequency response can therefore be written:

$$H(j\omega)=\frac{(1+j\omega/\omega_z)}{(1+j\omega/\omega_p)} \tag{24.62}$$

which is almost the same expression as in the third variant analysed above; the only thing that has changed is the low-frequency gain, which is now one.

Once more, to determine the amplitude and phase responses, it's most convenient to express this function in polar terms, which gives:

$$\begin{aligned}H(j\omega)=\frac{1+j\omega/\omega_z}{1+j\omega/\omega_p}&=\frac{\sqrt{1+\omega^2/\omega_z^{\,2}}\exp\left(j\tan^{-1}\left(\dfrac{\omega}{\omega_z}\right)\right)}{\sqrt{1+\omega^2/\omega_p^{\,2}}\exp\left(j\tan^{-1}\left(\dfrac{\omega}{\omega_p}\right)\right)}\\&=\frac{\sqrt{1+\omega^2/\omega_z^{\,2}}}{\sqrt{1+\omega^2/\omega_p^{\,2}}}\exp\left(j\left(\tan^{-1}\left(\frac{\omega}{\omega_z}\right)-\tan^{-1}\left(\frac{\omega}{\omega_p}\right)\right)\right)\end{aligned} \tag{24.63}$$

from which the amplitude response can be seen to be:

$$|H(j\omega)|=\sqrt{\frac{1+\omega^2/\omega_z^{\,2}}{1+\omega^2/\omega_p^{\,2}}} \tag{24.64}$$

which clearly starts at one at low frequencies, and decreases at very high frequencies to:

$$|H(\infty)|=\sqrt{\frac{1+\infty^2/\omega_z^{\,2}}{1+\infty^2/\omega_p^{\,2}}}=\sqrt{\frac{\infty^2/\omega_z^{\,2}}{\infty^2/\omega_p^{\,2}}}=\frac{\infty/\omega_z}{\infty/\omega_p}=\frac{\omega_p}{\omega_z} \tag{24.65}$$

So in some sense we have the opposite of the last circuit: it starts at a gain of one at low frequencies, and as the frequency increases the gain reduces to ω_p/ω_z (rather than the other way round).

The phase response hasn't changed, it is still given by:

$$\arg(H(j\omega))=\tan^{-1}\left(\frac{\omega}{\omega_z}\right)-\tan^{-1}\left(\frac{\omega}{\omega_p}\right) \tag{24.66}$$

but now since ω_p is at a lower frequency than ω_z, the second term will start to have a significant value at lower frequencies than the first term, and hence the phase difference between output and input would be expected to decrease before it increases again.

When plotted against the frequency on a logarithmic scale, these look like this (for the case shown the pole's break frequency is 1% of the zero's break frequency so ω_z / ω_p = 0.01):

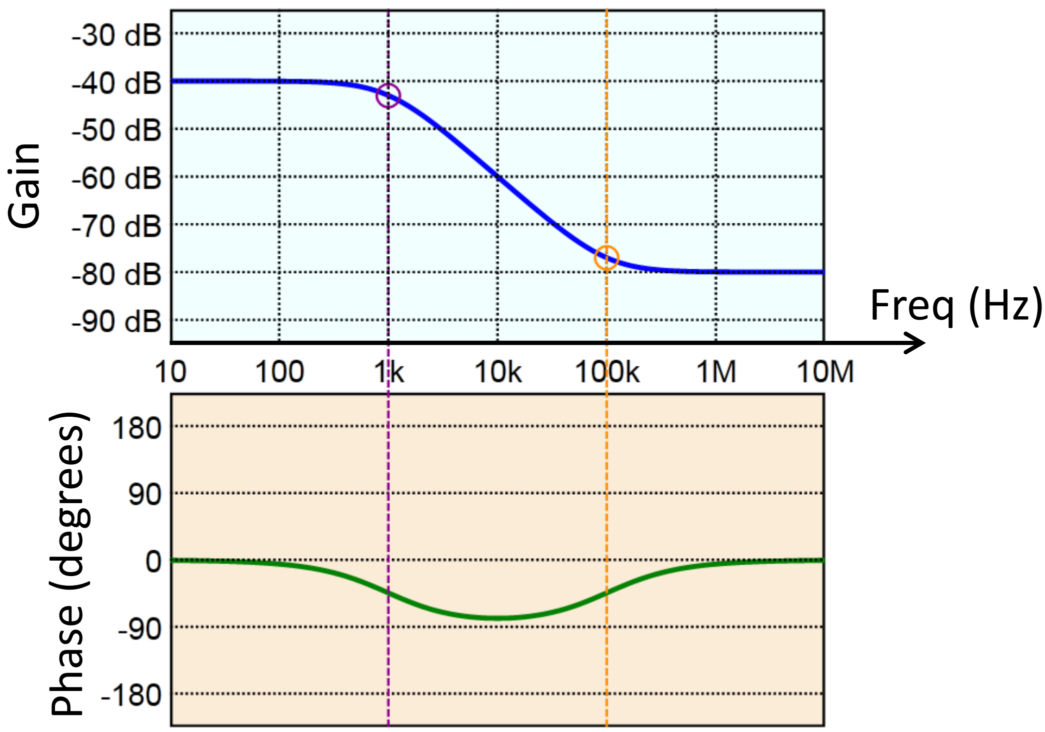

Figure 24.12 Amplitude and phase response of a non-zero zero and a lower frequency pole

Just like last time, at low frequencies the phase is zero, and at high frequencies the phase is zero as well; it's only in-between the zero and pole break frequencies, where the two terms in equation (24.39) do not cancel out, that the phase response of this circuit does anything interesting.

On to the step response. This time a nodal analysis gives:

$$V_{in} - V_{out}(t) = I_{R1}(t) \times R_1 \tag{24.67}$$

$$V_X(t) = I_{R2}(t) \times R_2 \tag{24.68}$$

$$I_C(t) = \frac{dQ(t)}{dt} = C\frac{d\left(V_{out}(t) - V_X(t)\right)}{dt} \tag{24.69}$$

$$I_{R2}(t) = I_C(t) = I_{R1}(t) \tag{24.70}$$

where I_{R1} is the current through R_1, I_{R2} is the current through R_2, I_C is the current through the capacitor, V_X is the voltage at the point marked 'X' on the circuit diagram, and Q is the charge on the top plate of the capacitor, and all of these are functions of time.

Noting that for all times of interest V_{in} = 1, and solving these for $V_{out}(t)$ reveals that:

$$V_{out}(t)\left(\frac{1}{R_1}\right) = \frac{1}{R_1} - C\left(1 + \frac{R_2}{R_1}\right)\frac{dV_{out}(t)}{dt} \tag{24.71}$$

Once again, I'll proceed with the same educated guess as before:

$$V_{out}(t) = A + B\exp\left(-\frac{t}{T}\right) \tag{24.72}$$

and substitute this into equation (24.71), then look for values of *A*, *B*, and *T* which work. This time the substitution gives:

$$\left(A + B\exp\left(-\frac{t}{T}\right)\right)\left(\frac{1}{R_1}\right) = \frac{1}{R_1} + \left(1 + \frac{R_2}{R_1}\right)\frac{CB}{T}\exp\left(-\frac{t}{T}\right) \tag{24.73}$$

and expanding terms gives:

$$\frac{A}{R_1} + \frac{B}{R_1}\exp\left(-\frac{t}{T}\right) = \frac{1}{R_1} + \left(1 + \frac{R_2}{R_1}\right)\frac{CB}{T}\exp\left(-\frac{t}{T}\right) \tag{24.74}$$

For this to be true at any time, the constant values have to be equal, so:

$$\begin{gathered} \frac{A}{R_1} = \frac{1}{R_1} \\ A = 1 \end{gathered} \tag{24.75}$$

and the coefficients of exp(-t/T) have to be equal, so:

$$\begin{aligned} \frac{B}{R_1} &= \frac{CB}{T}\left(1 + \frac{R_2}{R_1}\right) = \frac{CB}{T}\left(\frac{R_1 + R_2}{R_1}\right) \\ B &= \frac{CB}{T}\left(R_1 + R_2\right) \end{aligned} \tag{24.76}$$

From this we can deduce that the time-constant *T* must be:

$$T = C\left(R_1 + R_2\right) \tag{24.77}$$

but again we can't deduce a unique value for *B*. Again we have to look for a known point in the step response, and again the value at a time just after $t = 0$ is suitable; this value we can derive from the initial value theorem as:

$$H(\infty) = \frac{\left(1 - j\infty / z\right)}{\left(1 - j\infty / p\right)} = \frac{-j\infty / z}{-j\infty / p} = \frac{p}{z} \tag{24.78}$$

which here gives:

$$H(\infty) = \frac{p}{z} = \frac{\dfrac{-1}{\left(R_1 + R_2\right)C}}{\dfrac{-1}{R_2 C}} = \frac{R_2 C}{\left(R_1 + R_2\right)C} = \frac{R_2}{R_1 + R_2} \tag{24.79}$$

and therefore we can deduce a value for *B* from:

$$V_{out}\left(t=0_{+}\right)=A+B\exp\left(-\frac{0}{T}\right)=A+B$$
$$B=V_{out}\left(t=0_{+}\right)-A=\frac{R_2}{R_1+R_2}-1=\frac{-R_1}{R_1+R_2} \tag{24.80}$$

This makes the final time-domain solution:

$$V_{out}\left(t\right)=1-\frac{R_1}{R_1+R_2}\exp\left(-\frac{t}{T}\right) \tag{24.81}$$

This is a waveform which starts at $R_2 / (R_1 + R_2)$, and then increases to a final value of one with a time-constant of $(R_1 + R_2)C$. It looks like this:

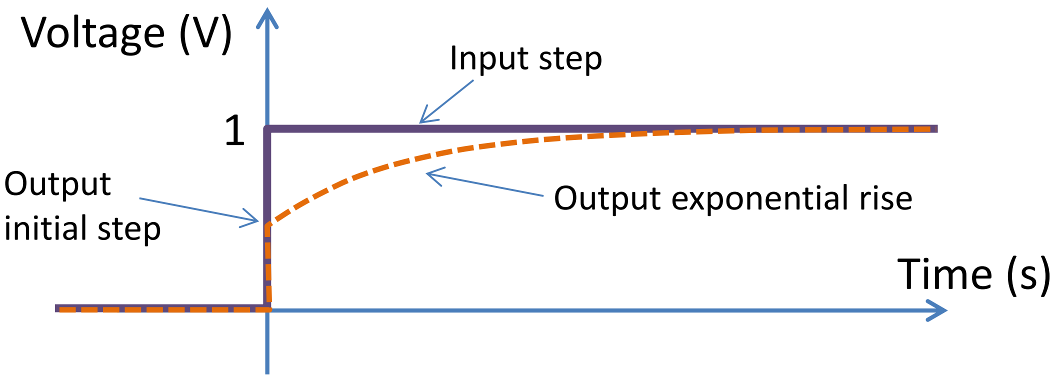

Figure 24.13 Step response of first-order system with one pole and a zero with a higher break frequency

We can use the final value theorem to check this result. The frequency response is:

$$H\left(j\omega\right)=\frac{\left(1-j\omega/z\right)}{\left(1-j\omega/p\right)} \tag{24.82}$$

and when the frequency is zero, this gives:

$$H\left(0\right)=\frac{\left(1-j0/z\right)}{\left(1-j0/p\right)}=1 \tag{24.83}$$

which is the expected final value as t becomes infinite, since:

$$V_{out}\left(t\right)=1-\frac{R_1}{R_1+R_2}\exp\left(-\frac{\infty}{T}\right)\to 1 \tag{24.84}$$

24.4 The short-cut to the step response

I've done the above analysis in the conventional, but perhaps slightly long-winded way: using the initial and final value theorems as checks that the solutions are correct.

You may be pleased to learn that there is another, much faster way to get the results for the step response: use the fact that a pole's break frequency is the inverse of the time-constant of an exponential decay, and use the initial and final value theorems to work out where that decay starts and stops.

The general form of a first-order step response is:

$$V_{out}(t) = A + B\exp\left(-\frac{t}{T}\right) \tag{24.85}$$

At time t = ∞, this is equal to the value of the frequency response at zero frequency, hence:

$$\begin{aligned} A + B\exp\left(-\frac{\infty}{T}\right) = A + 0 = H(0) \\ A = H(0) \end{aligned} \tag{24.86}$$

and time t = 0, this is equal to the value of the frequency response at infinite frequency, hence:

$$\begin{aligned} A + B\exp\left(-\frac{0}{T}\right) = A + B = H(\infty) \\ A + B = H(\infty) \\ B = H(\infty) - H(0) \end{aligned} \tag{24.87}$$

and the time constant T is minus one times the inverse of the pole p (in rad/s), so we could write the general form as:

$$V_{out}(t) = H(0) + \left(H(\infty) - H(0)\right)\exp(pt) \tag{24.88}$$

where p is the pole. With this method, once the frequency response is known, it's often possible to just write down the step response with no further algebra required.

24.5 Summary: the most important things to know

- First order responses have one pole in their frequency response
- Their step response can be written $V_{out}(t) = H(0) + \left(H(\infty) - H(0)\right)\exp(pt)$