21 A Short Introduction to Frequency Responses

v1.9 – June 2021

Prerequisite knowledge required: Phasors, AC Circuit Analysis

21.1 Introduction

Most of the interesting AC circuits have different gains at different frequencies. Sometimes this is what you want (in filter circuits that remove some frequencies or amplify others), sometimes this is not ideal, but nonetheless important to know about (for example in audio amplifiers). In both cases the frequency-dependency of the circuit can be described in terms of a *frequency response*.

Using phasors, it's possible to represent the frequency response of a circuit in terms of a complex function of frequency: the frequency response at any frequency is the ratio of the phasor representing the output waveform to the phasor representing the input waveform.

For example, in the previous note ("A Short Introduction to AC Circuit Analysis") we analysed the circuit shown below:

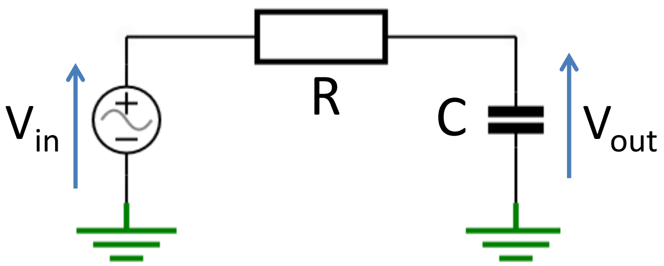

Figure 21-1 A simple RC low-pass filter circuit

and derived an expression for the phasor representation of the output voltage $\mathbf{V}_{\mathbf{out}}$ in terms of the phasor representation of the input voltage $\mathbf{V}_{\mathbf{in}}$:

$$\frac{\mathbf{V}_{\mathbf{out}}}{\mathbf{V}_{\mathbf{in}}} = \frac{1}{1 + j\omega RC} \tag{21.1}$$

This complex ratio has a magnitude (the voltage gain) but also a phase; the phase indicates a phase difference between the input and output, as illustrated below:

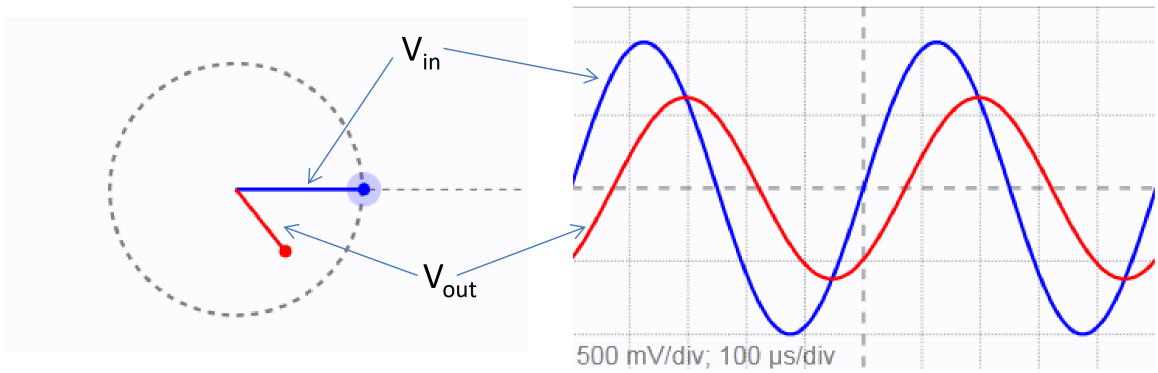

Figure 21-2 Phase and amplitude of response at one frequency

It turns out that deriving the frequency responses for any circuit formed from passive linear components (resistors, capacitors and inductors) always results in expressions that have the same general form, and which vary in frequency since they are function of $j\omega$ (usually written $H(j\omega)$).

$$H(j\omega) = \frac{\mathbf{V}_{\mathbf{out}}}{\mathbf{V}_{\mathbf{in}}} = \frac{\left(a_0 + a_1(j\omega) + a_2(j\omega)^2 + a_3(j\omega)^3 + \ldots\right)}{\left(b_0 + b_1(j\omega) + b_2(j\omega)^2 + b_3(j\omega)^3 + \ldots\right)} \tag{21.2}$$

in other words a polynomial in $j\omega$ divided by another polynomial in $j\omega$, and therefore that the entire response of the circuit (in terms of its amplitude response and phase response) can be specified with just a few coefficients. This is much easier than having to specify the gain and phase shift produced by the circuit at every frequency of interest.

This chapter is about why this happens, which parameters are usually specified when describing a frequency response (engineers don't use the coefficients a_0, a_1, b_0, b_1 etc.), and how to use the specified parameters to determine the gain and phase shift at any given frequency without doing a lot of complex calculations.

At this point I should introduce / remind readers of the concept of the *order* of a polynomial. The order of a polynomial is the highest power in the polynomial, so for example a quadratic is a polynomial of order two. It's also equal to the number of *roots* of the polynomial: values of the variable for which the polynomial is equal to zero. A polynomial of order n always has n roots, even although some of them may have the same value, for example the polynomial $x^2 - 2x + 1$ has two roots, although both of them occur when $x = 1$).

21.2 Plotting the frequency response: amplitude and phase responses

It's difficult to plot the entire frequency response against frequency on one graph (since the gain is complex), however the frequency response can be divided into two real responses, one describing how the amplitude changes with frequency (the *amplitude response*), and another describing how the phase difference between the input and output changes with frequency (the *phase response*).

For the example in Figure 21-1 this could be done as follows: first, to find the amplitude response, multiply equation (21.1) by its complex conjugate[1]:

$$\begin{aligned} |\mathbf{V}_{\mathbf{out}}|^2 &= |\mathbf{V}_{\mathbf{in}}|^2 \frac{1}{(1 + j\omega RC)} \frac{1}{(1 - j\omega RC)} \\ &= |\mathbf{V}_{\mathbf{in}}|^2 \frac{1}{1 + \omega^2 R^2 C^2} \end{aligned} \tag{21.3}$$

which gives:

$$\left| \frac{\mathbf{V}_{\mathbf{out}}}{\mathbf{V}_{\mathbf{in}}} \right| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} \tag{21.4}$$

[1] Using the useful fact that the product of a complex number and its complex conjugate is equal to the square of the amplitude of the magnitude of the number.

or if this were to be expressed in dB:

$$\begin{aligned} Gain(dB) &= -20\log_{10}\left(1+\omega^2 R^2 C^2\right)^{1/2} \\ &= -10\log_{10}\left(1+\omega^2 R^2 C^2\right) \end{aligned} \tag{21.5}$$

Then to find the phase response, the easiest approach is to express both the numerator and denominator in polar form:

$$\mathbf{V_{out}} = \mathbf{V_{in}} \frac{\exp(j0)}{\sqrt{1+\omega^2R^2C^2}\exp\left(j\tan^{-1}(\omega RC)\right)} \tag{21.6}$$

and then do the division:

$$\frac{\mathbf{V_{out}}}{\mathbf{V_{in}}} = \frac{1}{\sqrt{1+\omega^2R^2C^2}}\exp\left(0-j\tan^{-1}(\omega RC)\right) = \frac{1}{\sqrt{1+\omega^2R^2C^2}}\exp\left(j\tan^{-1}(-\omega RC)\right) \tag{21.7}$$

which suggests the phase of the output relative to the phase of the input is $\tan^{-1}(-\omega RC)$.

Alternatively, identify the real and imaginary parts of the frequency response by multiplying both numerator and denominator by the complex conjugate of the denominator, to get:

$$\frac{\mathbf{V_{out}}}{\mathbf{V_{in}}} = \frac{1}{(1+j\omega RC)}\frac{(1-j\omega RC)}{(1-j\omega RC)} = \frac{1-j\omega RC}{1+\omega^2 R^2 C^2} \tag{21.8}$$

$$\frac{\mathbf{V_{out}}}{\mathbf{V_{in}}} = \frac{1}{1+\omega^2 R^2 C^2} + j\frac{-\omega RC}{1+\omega^2 R^2 C^2} \tag{21.9}$$

and once the real and imaginary parts of a quantity are known, the argument (the phase difference in this case) can be determined using the inverse tangent function:

$$\arg\left(\frac{\mathbf{V_{out}}}{\mathbf{V_{in}}}\right) = \tan^{-1}\left(\frac{\text{Imag}}{\text{Real}}\right) = \tan^{-1}(-\omega RC) \tag{21.10}$$

This suggests that at very low frequencies there is no phase difference at all, but at very high frequencies there is a phase difference of around 90 degrees, with the output lagging the input (in other words being behind the input in phase) since $\tan^{-1}(-\infty) = -90$ degrees.

These responses can be plotted, and this is usually done using a *Bode plot* (a plot of the gain of a circuit in decibels and/or the phase in degrees against the log of the frequency). In this case, this reveals plots similar to those shown below.

In these plots, the gain as a function of frequency is a straight line at low frequencies and high frequencies. If extrapolated, these lines would meet at a frequency known as the *break frequency*. (The example below assumes the break frequency is 10 kHz.)

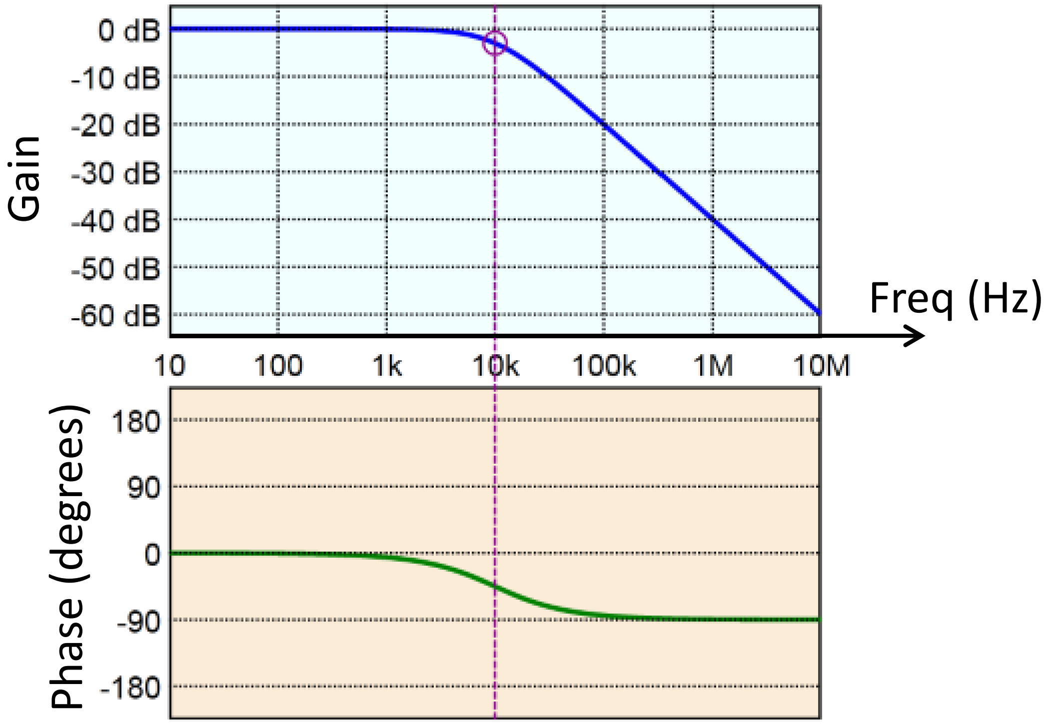

Figure 21-3 Amplitude and phase response of a simple series RC-network

There are a few interesting and important things to note about these plots:

- At frequencies below about one-tenth of the break frequency, the gain remains approximately constant at 0 dB, and the phase difference between output and input remains approximately constant at zero degrees.
- At frequencies above ten times the break frequency, the phase difference between output and input is approximately constant at 90 degrees.
- At frequencies above ten times the break frequency, the amplitude is decreasing at an approximately constant rate of 20 dB per decade.

These properties can be used to estimate the amplitude and phase response of more complex circuits, given knowledge of their gain at low frequencies, and the location and types of the break frequencies (see the chapter on "Bode Plots" for more details about how this is done).

21.3 Polynomials and roots, poles and zeros

As noted above, the frequency response of any circuit can be expressed as a ratio of two polynomials, and polynomials can be factorised in terms of their roots. For example, consider the numerator in equation (21.2) in the case where the maximum power of $j\omega$ is three (so this is a *third-order* polynomial):

$$\begin{aligned} a_0 + a_1(j\omega) + a_2(j\omega)^2 + a_3(j\omega)^3 &= a_3\left((j\omega)^3 + \frac{a_2}{a_3}(j\omega)^2 + \frac{a_1}{a_3}(j\omega) + \frac{a_0}{a_3}\right) \\ &= a_3(j\omega - z_0)(j\omega - z_1)(j\omega - z_2) \end{aligned} \tag{21.11}$$

where z_0, z_1 and z_2 are the three roots of the cubic polynomial in $j\omega$.

It's possible that the term a_0 is zero, and in this case the polynomial will have a root at zero (if both a_0 and a_1 are zero the polynomial will have two roots at zero, but this is less common in practice). In this situation, the polynomial could be factorised as:

$$\begin{aligned} 0+a_1(j\omega)+a_2(j\omega)^2+a_3(j\omega)^3 &= a_3(j\omega)\left((j\omega)^2+\frac{a_2}{a_3}(j\omega)+\frac{a_1}{a_3}\right) \\ &= a_3(j\omega)(j\omega-z_1)(j\omega-z_2) \end{aligned} \tag{21.12}$$

The denominator can also be factorised in this way, and for a cubic (third-order) polynomial with three roots this gives:

$$\begin{aligned} b_0+b_1(j\omega)+b_2(j\omega)^2+b_3(j\omega)^3 &= b_3\left((j\omega)^3+\frac{b_2}{b_3}(j\omega)^2+\frac{b_1}{b_3}(j\omega)+\frac{b_0}{b_3}\right) \\ &= b_3(j\omega-p_0)(j\omega-p_1)(j\omega-p_2) \end{aligned} \tag{21.13}$$

where p_0, p_1 and p_2 are the three roots of the cubic polynomial in $j\omega$. (It's impossible for a real frequency response to have a root of the denominator at zero Hz: this would mean that the circuit had an infinite gain at DC, and that's not stable.)

Combining these results, the general case of a frequency response can be written as:

$$H(j\omega)=\frac{\mathbf{V}_{\mathbf{out}}}{\mathbf{V}_{\mathbf{in}}}=\frac{a_3(j\omega-z_0)(j\omega-z_1)(j\omega-z_2)\ldots}{b_3(j\omega-p_0)(j\omega-p_1)(j\omega-p_2)\ldots} \tag{21.14}$$

Or at least it would in the case where there the numerator polynomial has no roots at zero; if there is one root at zero, then the expression would look like this:

$$H(j\omega)=\frac{\mathbf{V}_{\mathbf{out}}}{\mathbf{V}_{\mathbf{in}}}=\frac{a_3 j\omega(j\omega-z_1)(j\omega-z_2)}{b_3(j\omega-p_0)(j\omega-p_1)(j\omega-p_2)} \tag{21.15}$$

and if the numerator had two roots at zero, it would look like this:

$$H(j\omega)=\frac{\mathbf{V}_{\mathbf{out}}}{\mathbf{V}_{\mathbf{in}}}=\frac{a_3(j\omega)^2(j\omega-z_2)}{b_3(j\omega-p_0)(j\omega-p_1)(j\omega-p_2)} \tag{21.16}$$

and so on.

The roots of the numerator are the value of $j\omega$ at which the frequency response has a value of zero; these are known as the *zeros* of the response. The roots of the denominator are the values of $j\omega$ at which the frequency response has an infinite value; these are known as the *poles* of the response.

21.4 A general form for frequency responses

Equation (21.16) is not the usual form for expressing frequency responses. It is more common to express them in a slightly different way, making two changes:

1. Extract the factor of $-z_n$ or $-p_n$ from each bracketed term so that they take the form $-z_n\,(1 + j\omega / (-z_n))$ or $-p_n\,(1 + j\omega / (-p_n))$.
2. Combine all the constant terms outside the brackets into an overall term G.

This results in an equation for the frequency response that looks like this[2]:

$$\begin{aligned} H(j\omega) = \frac{\mathbf{V_{out}}}{\mathbf{V_{in}}} &= \frac{a_3(-z_0)(-z_1)(-z_2)\left(1+\dfrac{j\omega}{-z_0}\right)\left(1+\dfrac{j\omega}{-z_1}\right)\left(1+\dfrac{j\omega}{-z_2}\right)}{b_3(-p_0)(-p_1)(-p_2)\left(1+\dfrac{j\omega}{-p_0}\right)\left(1+\dfrac{j\omega}{-p_1}\right)\left(1+\dfrac{j\omega}{-p_2}\right)} \\ &= \frac{G\left(1+\dfrac{j\omega}{-z_0}\right)\left(1+\dfrac{j\omega}{-z_1}\right)\left(1+\dfrac{j\omega}{-z_2}\right)}{\left(1+\dfrac{j\omega}{-p_0}\right)\left(1+\dfrac{j\omega}{-p_1}\right)\left(1+\dfrac{j\omega}{-p_2}\right)} \end{aligned} \tag{21.17}$$

The reason for this last step is that many circuits of interest have a well-defined gain at DC (where all of the terms in brackets in the above expression evaluate to one), and this means that if G_{DC} is the gain at DC, we can write the general form like this:

$$H(j\omega) = \frac{\mathbf{V_{out}}}{\mathbf{V_{in}}} = \frac{G_{DC}\left(1+\dfrac{j\omega}{-z_0}\right)\left(1+\dfrac{j\omega}{-z_1}\right)\left(1+\dfrac{j\omega}{-z_2}\right)}{\left(1+\dfrac{j\omega}{-p_0}\right)\left(1+\dfrac{j\omega}{-p_1}\right)\left(1+\dfrac{j\omega}{-p_2}\right)} \tag{21.18}$$

and hence specify the performance of the circuit in terms of the gain at DC, and the terms z_n and p_n (the zeros and poles) only. This is easier to use, since the gain at DC is often an important parameter, and it's much easier to have this value directly in the equation.

There's just one problem with this: if there is a zero at a frequency of zero then this doesn't quite work, since G_{DC} (the gain at DC) would be zero. In this case the general form has to be left as:

$$H(j\omega) = \frac{\mathbf{V_{out}}}{\mathbf{V_{in}}} = \frac{G(j\omega)\left(1+\dfrac{j\omega}{-z_1}\right)\left(1+\dfrac{j\omega}{-z_2}\right)}{\left(1+\dfrac{j\omega}{-p_0}\right)\left(1+\dfrac{j\omega}{-p_1}\right)\left(1+\dfrac{j\omega}{-p_2}\right)} \tag{21.19}$$

where $j\omega G$ is the gain at low frequencies (frequencies much smaller than any of the values of the z and p terms, so that all of the other bracketed terms can be assumed to be insignificantly different from one).

If there are two zeros at 0 Hz, the frequency response is written:

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[2] You might be puzzled about why I've written the terms as $\left(1+\dfrac{j\omega}{-z}\right)$ rather than $\left(1-\dfrac{j\omega}{z}\right)$. The reason is that in practice, most of the zeros (and poles) are actually negative, so $-z$ is a positive number.

$$H(j\omega) = \frac{\mathbf{V}_{\mathbf{out}}}{\mathbf{V}_{\mathbf{in}}} = \frac{G(j\omega)^2\left(1+\dfrac{j\omega}{-z_2}\right)}{\left(1+\dfrac{j\omega}{-p_0}\right)\left(1+\dfrac{j\omega}{-p_1}\right)\left(1+\dfrac{j\omega}{-p_2}\right)} \tag{21.20}$$

where now the gain at frequencies much lower than any other pole or zero is given by $(j\omega)^2G = -\omega^2G$.

Extending these ideas to the case of multiple zeros at zero, leads to the general formula:

$$H(j\omega) = \frac{\mathbf{V}_{\mathbf{out}}}{\mathbf{V}_{\mathbf{in}}} = \frac{G(j\omega)^N\left(1+\dfrac{j\omega}{-z_N}\right)\left(1+\dfrac{j\omega}{-z_{N+1}}\right)\left(1+\dfrac{j\omega}{-z_{N+2}}\right)\ldots}{\left(1+\dfrac{j\omega}{-p_0}\right)\left(1+\dfrac{j\omega}{-p_1}\right)\left(1+\dfrac{j\omega}{-p_2}\right)\ldots} \tag{21.21}$$

where the terms z_i describe the effect of the zeros, and the terms p_i describe the effect of the poles, N is the number of zeros at zero, and $(j\omega)^N G$ is the gain at very low frequencies.

We have reduced the expression derived from circuit analysis to a few key parameters that can be used to determine the gain and phase difference between output and input at any frequency.

21.5 Poles, zeros and the coefficients in the general form

The existence of a pole in the frequency response results in a term $(1 + j\omega/(-p))$ in the denominator of the general form. It's useful to investigate what this term p actually represents in terms of the frequency response (and similarly what the terms z in the numerator represent).

Consider the total frequency response as the product of the gain at low frequencies, the effect of all of the zeros, and the effect of all of the poles. Something like:

$$H(j\omega) = \frac{\mathbf{V}_{\mathbf{out}}}{\mathbf{V}_{\mathbf{in}}} = G(j\omega)^N \times \left(1+\frac{j\omega}{-z_N}\right) \times \left(1+\frac{j\omega}{-z_{N+1}}\right) \times \frac{1}{\left(1+\dfrac{j\omega}{-p_0}\right)} \times \frac{1}{\left(1+\dfrac{j\omega}{-p_1}\right)}\ldots \tag{21.22}$$

Then consider expressing the amplitude of the frequency response in decibels:

$$\begin{aligned} 20\log_{10}\left(\left|H(j\omega)\right|\right) = {} & 20N\log_{10}\left(\left|G(j\omega)\right|\right) + 20\log_{10}\left(\left|1+\frac{j\omega}{-z_N}\right|\right) + 20\log_{10}\left(\left|1+\frac{j\omega}{-z_{N+1}}\right|\right) \\ & -20\log_{10}\left(\left|1+\frac{j\omega}{-p_0}\right|\right) - 20\log_{10}\left(\left|1+\frac{j\omega}{-p_0}\right|\right)\ldots \end{aligned} \tag{21.23}$$

In this form, the effect of all the poles and zeros add up to produce the final frequency response. Consider the contribution of just one pole:

$$Gain(dB) = -20\log_{10}\left(\left|1+\frac{j\omega}{-p}\right|\right) \tag{21.24}$$

(this is the term that results from a pole at $-p$). At very low frequencies, this can be well approximated by:

$$Gain(dB) = -20\log_{10}\left(\left|1+\frac{j\omega}{-p}\right|\right) \approx -20\log_{10}(1) = 0 \tag{21.25}$$

In other words at frequencies a lot lower than $|p|$, the pole has no effect on the frequency response (a similar result is true for zeros).

At high frequencies, the effect of the pole can be well approximated by:

$$Gain(dB) = -20\log_{10}\left(\left|1+\frac{j\omega}{-p}\right|\right) \approx -20\log_{10}\left(\left|\frac{j\omega}{-p}\right|\right) = -20\log_{10}\left(\frac{\omega}{|p|}\right) = +20\log_{10}\left(\frac{|p|}{\omega}\right) \tag{21.26}$$

$$Gain(dB) = 20\log_{10}(|p|) - 20\log_{10}(\omega) \tag{21.27}$$

which implies a loss which increases as the frequency increases[3], which when plotted on a graph of loss in dB against the log of the frequency would give a straight line (see Figure 21-4).

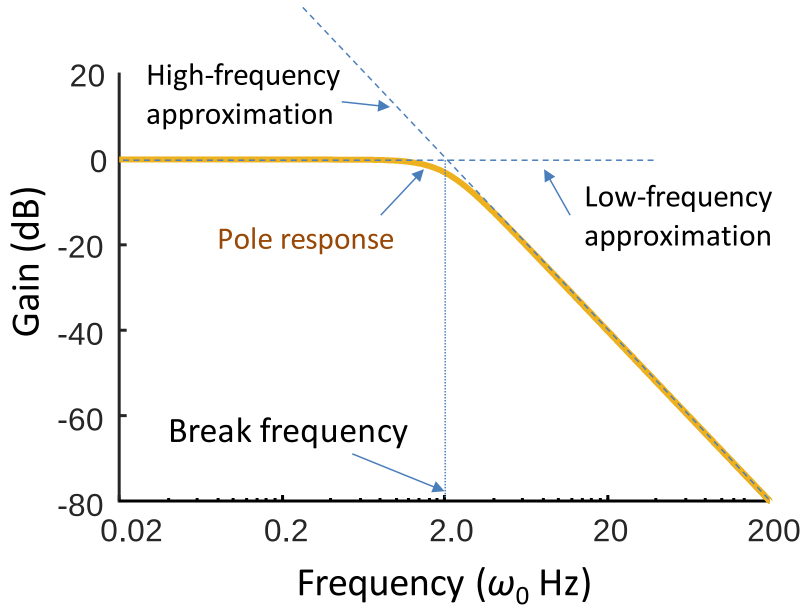

Figure 21-4 Low and high frequency approximations to the response of a pole at 2 rad/s

Plotting both of these approximations on a graph of dB against the logarithm of frequency results in straight lines. Where these straight lines meet is given by:

[3] Notice that I've used the modulus of the pole |p|, but not the modulus of the angular frequency ω in equation (21.27). This is because the frequency here is always real and positive so I don't need to worry about taking the modulus. The pole however can be complex or negative, so I need to keep the modulus sign here.

$$
\begin{aligned}
-20\log_{10}\left(\left|\frac{\omega}{p}\right|\right) &= 0 \\
\log_{10}\left(\left|\frac{\omega}{p}\right|\right) &= 0 \\
\left(\frac{\omega}{|p|}\right) &= 1 \\
\omega &= |p|
\end{aligned}
\tag{21.28}
$$

In other words, the modulus of the term *p* in the general form of the frequency response is the frequency at which the low-frequency straight-line approximation and the high-frequency straight-line approximation meet. This is known as the *break frequency* of the pole (sometimes also called the *corner frequency* or the *cut-off frequency*).

A similar result holds for zeros: the modulus of the terms *z* in the general form are the frequencies at which the low-frequency approximation and the high-frequency approximation to the effect of the zero meet. This is known as the *break frequency* of the zero.

21.5.1 Poles, break frequencies and units

There is often some confusion about poles, their units, and their effects. If you're getting confused here, the following summary / discussion might help (if you're not confused, feel free to skip this bit):

- Poles and zeros are the roots of the denominator and numerator polynomials in $j\omega$ in the frequency response.
- Since poles and zeros are values of $j\omega$, you can think of them as complex frequencies. They have the same units as frequency, although they do not correspond to oscillations.
- A real pole (e.g. at -2 rad/s) means that the frequency response become infinite when $j\omega$ is minus two. If $j\omega = -2$, then $\omega = 2j$. That's an imaginary frequency.
- An cisoidal oscillation with a frequency of *v* rad/s can be interpreted in the time domain as:

  $$\exp(j\nu t) \tag{21.29}$$

  which suggests that a cisoidal oscillation with a frequency of 2j can be interpreted as:

  $$\exp(j2jt) = \exp(j^2 2t) = \exp(-2t) \tag{21.30}$$

  That is an exponential decay. This is a much better interpretation of what a pole at -2 represents in the time-domain.
- While poles and zeros are not frequencies (in the sense of a frequency being the inverse of the period of an oscillation), there is a real frequency associated with them: this is known as the break frequency. The break frequency is where the low-frequency asymptote intersects with the high-frequency asymptote in the amplitude response of the pole or zero. This

break frequency turns out to be equal to the modulus of the pole or zero. The break frequency really is a frequency.

- Sometimes, I might be lazy and refer to the "pole frequency" or "the pole at 10 kHz". By this, I am referring to the modulus of the pole: the distance of the pole from the origin of the Argand diagram. This is numerically equal to the break frequency. (The same is true for zeros.) Please let me know if I do this; it's a bad habit and I'm trying to give it up.

21.6 A few simple examples

The frequency response for the circuit in Figure 21-1 has previously been calculated to be:

$$H(j\omega) = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = \frac{1}{1 + j\omega RC} \tag{21.31}$$

To express this in terms of a low-frequency gain and the break frequencies of the poles and zeros, we have to get this into the form of the general formula:

$$H(j\omega) = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = \frac{G(j\omega)^N \left(1 + \frac{j\omega}{-z_N}\right)\left(1 + \frac{j\omega}{-z_{N+1}}\right)\left(1 + \frac{j\omega}{-z_{N+2}}\right)\ldots}{\left(1 + \frac{j\omega}{-p_0}\right)\left(1 + \frac{j\omega}{-p_1}\right)\left(1 + \frac{j\omega}{-p_2}\right)\ldots} \tag{21.32}$$

The numerator and denominator of equation (21.31) are already in the form of polynomials, with the numerator having order zero (in other words no terms in $j\omega$) and the denominator order one (the highest power of $j\omega$ is one). The general expression with no zeros and one pole is therefore:

$$H(j\omega) = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = \frac{G}{\left(1 + \frac{j\omega}{-p_0}\right)} \tag{21.33}$$

and equating this to equation (21.31) reveals that we need to find G and p_0 such that:

$$\frac{G}{\left(1 + \frac{j\omega}{-p_0}\right)} = \frac{1}{1 + j\omega RC} \tag{21.34}$$

If $G = 1$ and $p_0 = -1/RC$ then these two expressions are the same. We can therefore conclude that this circuit has a low-frequency gain of one, and a single pole with break frequency at $1/RC$ rad/s.

21.6.1 A slightly more difficult circuit

Suppose the resistor and the capacitor in this circuit were swapped round:

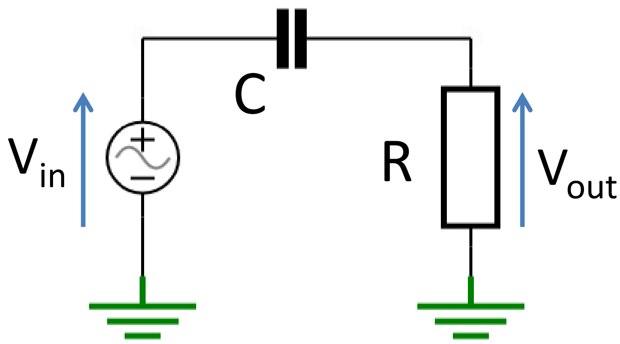

Figure 21-5 A single-pole circuit with a zero at zero Hz

Following through the AC circuit analysis of this circuit leads to the frequency response:

$$H(j\omega) = \frac{\mathbf{V}_{out}}{\mathbf{V}_{in}} = \frac{j\omega RC}{1 + j\omega RC} \tag{21.35}$$

Now there is a zero at zero (the numerator is zero when the angular frequency ω is zero), so we have to use the form of the general equation with one zero at zero, and one pole:

$$\frac{G(j\omega)}{\left(1 + \dfrac{j\omega}{-p_0}\right)} = \frac{j\omega RC}{1 + j\omega RC} \tag{21.36}$$

and here the two expressions for the frequency response can be made equal by setting $p_0 = -1/RC$ and $G = RC$. This circuit could be described as having a zero at zero, a pole with a break frequency of 1/RC rad/s and a gain at low frequencies of $j\omega RC$.

21.6.2 A more interesting circuit

For a slightly more complex example, what about the circuit shown below?

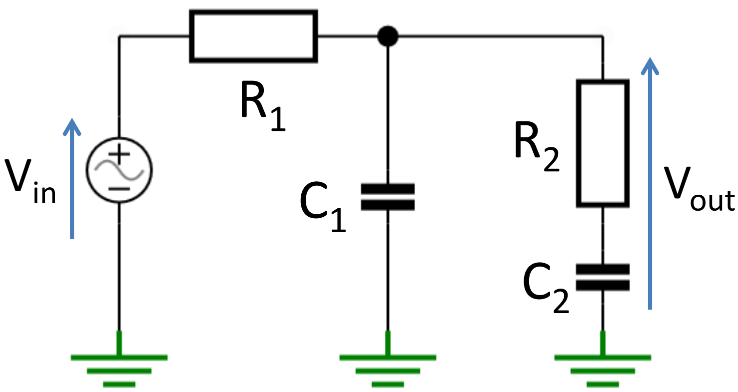

Figure 21-6 A slightly more complex circuit

Once again we have a potential divider, but this time with a single resistor (R_1) on “the top” and a network containing the parallel combination of a capacitor and a resistor and capacitor in series on “the bottom”.

Using complex impedances we could write the output in terms of the input as:

$$\mathbf{V}_{\mathbf{out}} = \mathbf{V}_{\mathbf{in}} \frac{1/j\omega C_1 \times (\mathrm{R}_2 + 1/j\omega C_2)/(1/j\omega C_1 + 1/j\omega C_2 + R_2)}{R_1 + 1/j\omega C_1 \times (\mathrm{R}_2 + 1/j\omega C_2)/(1/j\omega C_1 + 1/j\omega C_2 + R_2)} \tag{21.37}$$

which after a large amount of particularly tedious algebra reveals that:

$$\mathbf{V}_{\mathbf{out}} = \mathbf{V}_{\mathbf{in}} \frac{1 + j\omega R_2 C_2}{1 + j\omega(R_1 C_1 + R_1 C_2 + R_2 C_2) - \omega^2 R_1 R_2 C_1 C_2} \tag{21.38}$$

This is clearly a polynomial of order one in the numerator (and hence there is one zero in the response) and a polynomial of order two (a quadratic) in the denominator (and hence there are two poles in the response).

To identify the gain at low frequency and the break frequencies of the poles and zeros, we need to get this into the form:

$$H(j\omega) = \frac{\mathbf{V}_{\mathbf{out}}}{\mathbf{V}_{\mathbf{in}}} = \frac{G\left(1 + \frac{j\omega}{-z_0}\right)}{\left(1 + \frac{j\omega}{-p_0}\right)\left(1 + \frac{j\omega}{-p_1}\right)} \tag{21.39}$$

Or in other words, find G, z_0, p_0 and p_1 such that:

$$\frac{1 + j\omega R_2 C_2}{1 + j\omega(R_1 C_1 + R_1 C_2 + R_2 C_2) - \omega^2 R_1 R_2 C_1 C_2} = \frac{G\left(1 + \frac{j\omega}{-z_0}\right)}{\left(1 + \frac{j\omega}{-p_0}\right)\left(1 + \frac{j\omega}{-p_1}\right)} \tag{21.40}$$

Multiplying out the brackets on the bottom of the right-hand side of this equation gives:

$$\frac{1 + j\omega R_2 C_2}{1 + j\omega(R_1 C_1 + R_1 C_2 + R_2 C_2) - \omega^2 R_1 R_2 C_1 C_2} = \frac{G\left(1 + \frac{j\omega}{z_0}\right)}{1 + j\omega\left(\frac{1}{-p_0} + \frac{1}{-p_1}\right) - \omega^2 \frac{1}{p_0 p_1}} \tag{21.41}$$

from which we can determine that:

$$\begin{aligned} &G = 1 && z_0 = 1/R_2 C_2 \\ &\frac{1}{p_0} + \frac{1}{p_1} = -(R_1 C_1 + R_1 C_2 + R_2 C_2) && \frac{1}{p_0 p_1} = R_1 R_2 C_1 C_2 \end{aligned} \tag{21.42}$$

To work out the actual break frequencies of the poles now requires the solution of a quadratic equation, and in this case, the poles turn out to be at:

$$p_{1,2} = \frac{-(R_1 C_1 + R_1 C_2 + R_2 C_2) \pm \sqrt{(R_1 C_1 + R_1 C_2 + R_2 C_2)^2 + 4 R_1 R_2 C_1 C_2}}{2 R_1 R_2 C_1 C_2} \tag{21.43}$$

and as before the break frequencies are the moduli of the values of the poles.

In this case the roots of the quadratic are real, since the square root required is always the square root of a positive number. But of course, quadratic equations don't always have real roots…

21.6.3 Complex poles and zeros

Consider the circuit shown below:

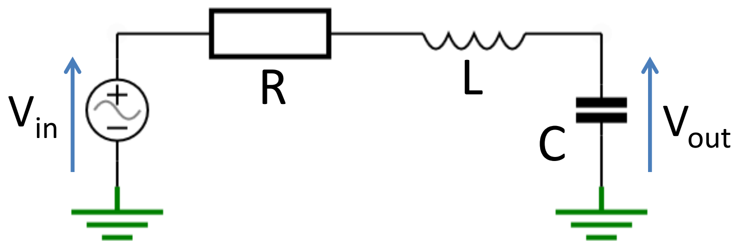

Figure 21-7 A circuit with complex poles

Follow through the standard AC analysis for this circuit, and the result is:

$$H(j\omega) = \frac{\mathbf{V}_{\mathbf{out}}}{\mathbf{V}_{\mathbf{in}}} = \frac{1/j\omega C}{R + j\omega L + 1/j\omega C} = \frac{1}{1 + j\omega RC + (j\omega)^2 LC} \tag{21.44}$$

and once again there is a quadratic in the denominator that requires to be solved to find the location of the poles. However this time, the poles turn out to be at:

$$p_{0,1} = -\frac{R}{2L} \pm \frac{\sqrt{R^2 - 4L/C}}{2L} \tag{21.45}$$

It's perfectly possible for the term $4L/C$ to be larger than R^2, and if this happens, the square root required would result in an imaginary answer, and hence lead to a pole with a complex value.

That's no problem at all. Well, it does make things a little harder to work out, but it's no problem for the maths or the physics of the situation. Poles (and zeros) really can be complex, although their break frequencies will still be real (the break frequency is still the modulus of the value of the pole or zero).

Where this does cause some additional difficulties is when trying to use the poles and zeros to predict the frequency response of the system, particularly around the break frequencies where neither the low-frequency approximation or high-frequency approximation give an accurate estimate of the effect of the pole or zero.

These cases really need more thought. Firstly, note that since the coefficients of the quadratic equation are real, any complex poles will appear as a complex conjugate pair. In fact, they are often left as a pair, and described by two other quantities that are simpler to work out than the poles themselves (the resonant frequency and the Q-factor[4]).

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[4] I should probably mention that this isn't the only way of doing this, and you might find some textbooks and other sources talking in terms of the resonant frequency and the damping ratio instead of the resonant

The resonant frequency (ω_0) and Q-factor (Q) for a pair of poles (or zeros) are defined by the equation:

$$\left(1+\frac{j\omega}{-p_0}\right)\left(1+\frac{j\omega}{-p_1}\right)=1+j\omega\left(\frac{1}{Q\omega_0}\right)+\frac{(j\omega)^2}{\omega_0{}^2} \tag{21.46}$$

Expressing the frequency response in these terms, it's much easier to start from:

$$\frac{1}{Q\omega_0}=RC \qquad \frac{1}{\omega_0{}^2}=LC \tag{21.47}$$

and work out that:

$$Q=\frac{1}{R}\sqrt{\frac{L}{C}} \qquad \omega_0{}^2=\frac{1}{LC} \tag{21.48}$$

than it is to solve the quadratic equation, so this is what's usually done. An engineer is much more likely to refer to a pair of poles as having a "resonant frequency of X and a Q-factor of Y" than to describe the poles in terms of their real and imaginary components. For more about the effect of these complex poles on the frequency response, see the chapter on "Second-Order Responses".

21.7 High and low frequency limiting cases

There's often a short-cut (or at least a simple way of checking the maths) when deriving the gains, poles and zeros of real circuits, and that's to consider their behaviour at both high and low frequencies.

Look at Figure 21-1 again. At low frequencies, the capacitor has infinite impedance, so we would expect the output to be equal to the input. In other words, the gain should be one, and the phase shift between the input and output should be zero. Then look at the form of the frequency response:

$$H(j\omega)=\frac{\mathbf{V}_{\mathbf{out}}}{\mathbf{V}_{\mathbf{in}}}=\frac{1}{1+j\omega RC} \tag{21.49}$$

and set the angular frequency to zero. This gives:

$$H(j\omega)=\frac{\mathbf{V}_{\mathbf{out}}}{\mathbf{V}_{\mathbf{in}}}=\frac{1}{1+j0RC}=\frac{1}{1+0}=1 \tag{21.50}$$

which as expected gives a gain of one and no phase shift.

At high frequencies, where the terms in $j\omega/z_n$ and $j\omega/p_n$ are much greater than one, the frequency response in this case can be well approximated by:

---

frequency and the Q-factor. The damping ratio is 1/2Q (or alternatively, the Q-factor is half the inverse of the damping factor).

$$H(j\omega)=\frac{\mathbf{V}_{\mathbf{out}}}{\mathbf{V}_{\mathbf{in}}}=\frac{1}{1+j\omega RC}\approx\frac{1}{j\omega RC}=\frac{-j}{\omega RC}=\frac{1}{\omega RC}\exp\left(-j\frac{\pi}{2}\right) \quad (21.51)$$

which shows that the output is lagging the input by 90 degrees ($\pi/2$ radians) and the gain is proportional to the inverse of the frequency.

Similar approximations can be applied to more complex expression, including the general form:

$$H(j\omega)=\frac{\mathbf{V}_{\mathbf{out}}}{\mathbf{V}_{\mathbf{in}}}=\frac{G(j\omega)^N\left(1+\frac{j\omega}{-z_N}\right)\left(1+\frac{j\omega}{-z_{N+1}}\right)\left(1+\frac{j\omega}{-z_{N+2}}\right)\dots}{\left(1+\frac{j\omega}{-p_0}\right)\left(1+\frac{j\omega}{-p_1}\right)\left(1+\frac{j\omega}{-p_2}\right)\dots} \quad (21.52)$$

At very low frequencies, all of the bracketed addition terms evaluate to approximately one (since all terms like $j\omega/p$ are much smaller than one), and this becomes:

$$H(j\omega)=\frac{\mathbf{V}_{\mathbf{out}}}{\mathbf{V}_{\mathbf{in}}}=G(j\omega)^N=G\omega^N j^N=G\,\omega^N\exp\left(j\frac{N\pi}{2}\right) \quad (21.53)$$

which has a gain proportional to the frequency raised to the power of the number of zeros at zero, and an output which is ahead of the input in phase by 90 degrees times the number of zeros at zero Hz.

At very high frequencies, the approximation becomes:

$$H(j\omega)=\frac{\mathbf{V}_{\mathbf{out}}}{\mathbf{V}_{\mathbf{in}}}=\frac{G(j\omega)^N\left(\frac{j\omega}{-z_N}\right)\left(\frac{j\omega}{-z_{N+1}}\right)\left(\frac{j\omega}{-z_{N+2}}\right)\dots}{\left(\frac{j\omega}{-p_0}\right)\left(\frac{j\omega}{-p_1}\right)\left(\frac{j\omega}{-p_2}\right)\dots}=\frac{G(j\omega)^{NZ-NP}(-p_0)(-p_1)(-p_2)\dots}{(-z_N)(-z_{N+1})(-z_{N+2})\dots} \quad (21.54)$$

where *NZ* is the total number of zeros (including the zeros at zero Hz) and *NP* is the total number of poles, and this is often a good way of determining *G* for systems that contain one or more zeros at zero (for example if their gain at high frequencies is known).

21.8 Poles and zeros on the Argand diagram

Since poles and zeros can be complex, there's a convenient way to represent poles and zeros on the Argand diagram. For example, a circuit with a single pole with a break frequency at 1 kHz could be represented on an Argand diagram as a point at co-ordinates (-2000π, 0), as shown below:

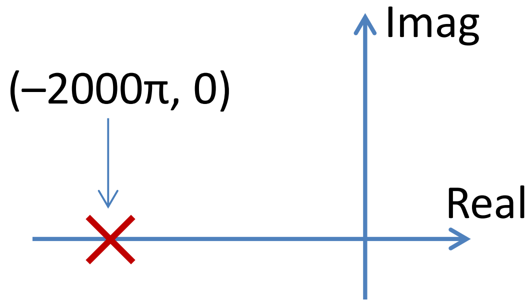

Figure 21-8 A pole shown on the Argand diagram

This is useful, since it gives a geometric way of calculating the effect of the pole at any frequency. Put a point at $j\omega$, and draw the line from that point to the pole:

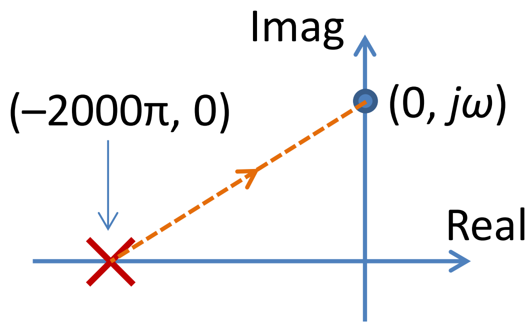

Figure 21-9 A pole and frequency shown on the Argand diagram

This line represents the difference between the point (0, jω) and the point (-2000π, 0), so the length and direction of this line could be represented as the complex number *L* where:

$$\begin{aligned} -2000\pi + L &= j\omega \\ L &= j\omega + 2000\pi \end{aligned} \tag{21.55}$$

which is proportional to:

$$\left(1 + \frac{j\omega}{2000\pi}\right) \tag{21.56}$$

and this is the corresponding term in the frequency response equation.

In other words, the length of the line from a pole to a point on the imaginary axis corresponding to any given frequency is proportional to the gain of the circuit due to that pole, and the direction of the line gives the phase difference between the output and input of the circuit due to the pole.

It works for zeros as well:

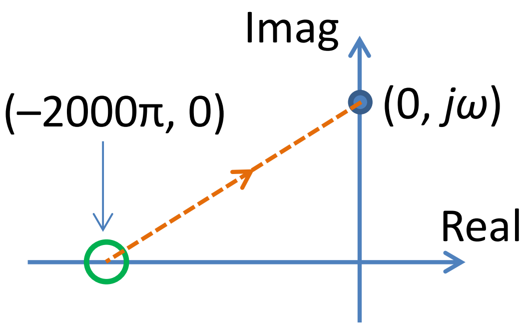

Figure 21-10 A zero and frequency shown on the Argand diagram

In the case where there are multiple poles and zeros, the total effect on the gain and phase difference between the output and input of the circuit can be determined by just combining the lengths and phases of all of these lines.

For example, consider a circuit with a pole at 1 kHz and a zero at 10 kHz:

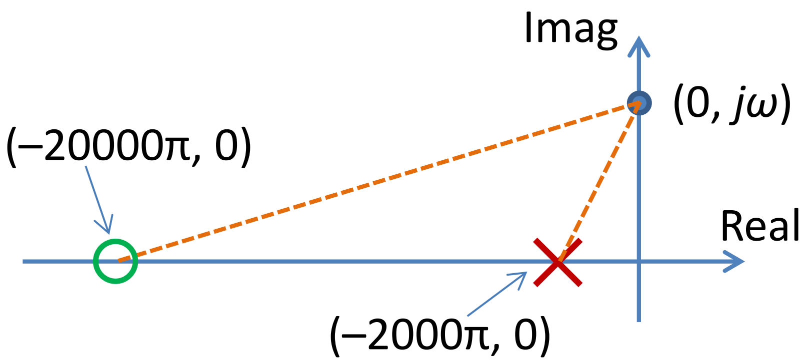

Figure 21-11 A zero and frequency shown on the Argand diagram

At low frequencies, the effect of the pole is greater than the effect of the zero, since the line from the zero to the frequency will not be changing in length or direction as quickly. However for very large frequencies, the length of the lines from the pole and the zero to the frequency point on the imaginary axis will be approximately the same, which suggests that the gain of the circuit will be almost constant with frequency as the effects of the pole and the zero will cancel each other out.

This is indeed what happens.

21.8.1 Complex poles on the Argand diagram

The technique of thinking of poles and zeros as being points on an Argand diagram is also useful when considering the effect of complex poles. Suppose there is a pair of complex poles in the response of a circuit, as shown in the figure below; the poles are a distance of R from the origin, at an angle of θ from the negative real axis.

Now, as the frequency increases from zero, the distance from the frequency point to the nearest pole is actually decreasing (this never happens when the pole is on the real axis). Decreasing the distance to a pole increases the gain due to that pole, and with poles close to the imaginary axis, this can result in a peak in the frequency response around the pole frequency. (For more details about this effect, see the chapter on "Second-Order Frequency Responses".)

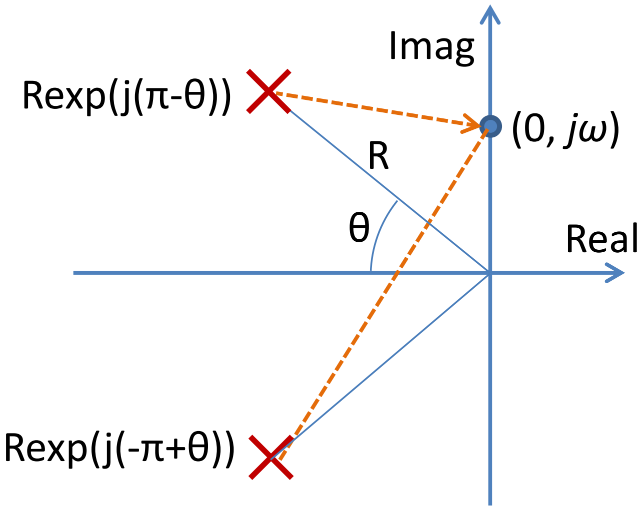

Figure 21-12 A pair of complex poles on the Argand diagram

Finally in this section, it's worth noting the relationship between the location of the poles in terms of polar co-ordinates and the resonant frequency and Q-factor of these poles.

The poles here could be expressed in terms of their polar co-ordinates as:

$$p_0 = \mathrm{R}\exp\left(j\left(\pi-\theta\right)\right) \qquad\qquad p_1 = \mathrm{R}\exp\left(j\left(\pi+\theta\right)\right) \tag{21.57}$$

which when substituting into the denominator of the frequency response equation gives:

$$\begin{aligned}\left(1+\frac{j\omega}{-p_0}\right)\left(1+\frac{j\omega}{-p_1}\right) &= 1+j\omega\left(\frac{1}{-p_0}+\frac{1}{-p_1}\right)+\left(j\omega\right)^2\frac{1}{p_0p_1} \\ &= 1+\frac{j\omega}{R}\left(\frac{1}{\exp\left(j\theta\right)}+\frac{1}{\exp\left(-j\theta\right)}\right)+\left(j\omega\right)^2\frac{1}{R^2\exp\left(j2\pi\right)} \\ &= 1+\frac{j\omega}{R}\left(\exp\left(j\theta\right)+\exp\left(-j\theta\right)\right)+\frac{\left(j\omega\right)^2}{R^2} \\ &= 1+\frac{j\omega}{R}\left(2\cos\left(\theta\right)\right)+\frac{\left(j\omega\right)^2}{R^2}\end{aligned} \tag{21.58}$$

Compare this to the standard form in terms of the resonant frequency and Q-factor:

$$1+\frac{j\omega}{R}\left(2\cos\left(\theta\right)\right)+\frac{\left(j\omega\right)^2}{R^2} = 1+\frac{j\omega}{Q\omega_0}+\frac{\left(j\omega\right)^2}{\omega_0{}^2} \tag{21.59}$$

and we can readily see from comparing coefficients of $(j\omega)^2$ that:

$$\omega_0 = R \tag{21.60}$$

and from comparing the coefficients of $j\omega$ that:

$$
\begin{aligned}
Q\omega_0 &= \frac{R}{2\cos(\theta)} \\
Q &= \frac{R}{2\omega_0\cos(\theta)} = \frac{R}{2R\cos(\theta)} = \frac{1}{2\cos(\theta)}
\end{aligned} \tag{21.61}
$$

So in terms of the location of the poles on the Argand diagram, the resonant frequency of a pair of complex poles is the length of the line from the origin to those poles, and the Q-factor is half of the inverse of the cosine of the angle between the poles and the negative real axis[5].

21.9 Summary: the most important things to know

- The frequency response of a network $H(j\omega)$ is the ratio of the phasor representing the output to the phasor representing the input.
- The frequency response of almost all circuits can be written in the general form:

$$
H(j\omega) = \frac{\mathbf{V}_{\mathbf{out}}}{\mathbf{V}_{\mathbf{in}}} = \frac{G(j\omega)^N\left(1+\frac{j\omega}{-z_N}\right)\left(1+\frac{j\omega}{-z_{N+1}}\right)\left(1+\frac{j\omega}{-z_{N+2}}\right)\ldots}{\left(1+\frac{j\omega}{-p_0}\right)\left(1+\frac{j\omega}{-p_1}\right)\left(1+\frac{j\omega}{-p_2}\right)\ldots}
$$

  where $G(j\omega)^N$ is the gain at low frequencies, the z_n are called zeros and the p_n are called poles.
- The *break frequency* of a pole or zero is the modulus of the pole or zero.
- Well below the break frequency, poles and zeros don't have much effect on the circuit. Well above the break frequency, poles cause the amplitude response to drop at 20 dB/decade, and zeros cause the amplitude response to rise at 20 dB/decade.
  - "Well below" and "well above" can be taken to be a factor of ten below and a factor of ten above respectively.
- Systems with two poles (or two zeros) can be specified in terms of the resonant frequency and Q-factor rather than the values of the poles (or zeros).
  - The resonant frequency is the geometric mean of the poles
  - The Q-factor is the resonant frequency divided by the modulus of the sum of the poles.
- Poles and zeros can be complex.

[5] If you're reading the footnotes, you might remember an earlier one which mentioned the *damping factor*, and noted that the damping factor was $1/2Q$. This is why the damping factor is defined in this way: it just becomes equal to cos(θ), which is a simple way of visualising it.