20 A Short Introduction to AC Circuit Analysis

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Prerequisite knowledge required: Phasors, Complex Impedance, Ohm and Kirchhoff's Laws

20.1 Introduction

While both of Kirchhoff's laws can still be used with AC circuits, the third cornerstone of DC circuit analysis (Ohm's law) cannot be applied directly to AC circuits, since capacitors and inductors do not obey Ohm's law: the current through these components at every instant is not proportional to the voltage across them.

However, it turns out that with a few mathematical tricks, a version of Ohm's law can be used with capacitors and inductors in AC circuits to give a good approximation to the behaviour of these circuits in many situations[1]. Once these tricks are known, analysis of linear AC circuits is just as straightforward as the analysis of DC circuits.

This note attempts to illustrate the use of these tricks, and show how AC circuits can be analysed using complex[2] versions of Kirchhoff and Ohm's Laws.

20.2 Oscillations and phase

A reminder from earlier chapters: any sinusoidal oscillation (and for the moment we're only dealing with sinusoidal oscillations) can be represented by the equation:

$$x(t) = A\cos(\omega t + \theta) \tag{20.1}$$

and such an oscillation can also be represented in phasor terms as a vector with length A at an angle of θ to the horizontal, or as a complex number with an amplitude of A and an argument of θ. In a linear circuit, having a single-frequency sinusoidal input implies that all the voltages and currents in the circuit are also single-frequency sinusoids with the same frequency (albeit with different amplitudes and phases).

The subject of AC circuit analysis is concerned with working out the amplitudes and the phases of all the voltages and currents in the circuit. This is slightly more complicated than the equivalent problem in DC circuit analysis, since it's now not just the amplitudes (the maximum value of the currents and voltages) that needs to be determined, it's the relative phases as well.

For example, the three phasors in the following diagram represent the three voltages in a potential divider circuit. The green signal is the input, and the blue and red signals (the smaller two) are the voltages across the two components in the potential divider.

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[1] You'll note a few "weasel words" there: "good approximation" and "in many situations". The techniques I'm about to describe don't always work, and it's important to know about these exceptions; more about this later.

[2] That's "complex" in the sense of complex numbers, not "complex" in the sense of complicated. It's not that complicated really. I wish they'd chosen a different name for complex numbers, one that doesn't put people off so much.

You might notice that although at every instant, the red signal plus the blue signal is equal to the green signal, it is not true to say that the amplitude of the red signal plus the amplitude of the blue signal is equal to the amplitude of the green signal. That's hopefully obvious from the diagram below.

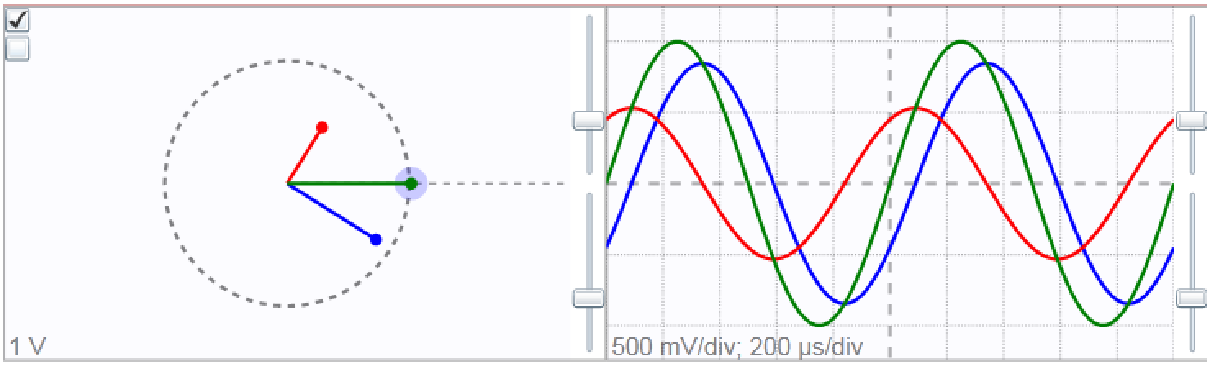

Figure 20.1 Phasor diagram of the voltages in a potential divider

Once you've accepted and understood that fact (the sum of two AC voltages does not produce a signal with the sum of the amplitudes of the voltages) you've understood the most fundamental issue in analysing AC circuits. It's worth thinking about this one until it becomes clear before moving on; it's probably the most common mistake made when starting to learn AC circuit analysis.

20.3 Resistors and AC circuits

Resistors are easy. Resistors obey Ohm's law, which states that:

$$V = I \times R \tag{20.2}$$

In other words: the potential difference between the terminals at each end of the resistor is equal to the product of the resistance and the current flowing through the resistor. This applies to every instant of time, so the voltage across a resistor is always equal to the resistance times the current flowing through the resistor.

This implies that the maximum value (the amplitude) of the voltage must occur at the same time as the maximum value (the amplitude) of the current: in other words the voltage and current have the same phase (they are *in phase*). This also means that on a phasor diagram, the phasor representing the voltage across an ideal resistor will always be in the same direction as the phasor representing the current flowing through it.

So that's fine.

20.4 Capacitors and AC circuits

Capacitors are not so easy. Capacitors obey the equation:

$$Q = C \times V \tag{20.3}$$

The charge on one plate of a capacitor is equal to the product of the capacitance and the voltage across the capacitor. Taking just one complex sinusoidal frequency[3], and we could write:

$$Q(t) = Q_{max} \exp(j\omega t) = C \times V_{max} \exp(j\omega t) \tag{20.4}$$

where Q_{max} is the maximum charge and V_{max} is the maximum voltage.

Then we could note that the rate of change of the charge is equal to the current: after all if the charge is increasing, then current must be flowing onto one plate (and off the other plate) of the capacitor. We can differentiate equation (20.4) to find the rate at which charge is moving onto the capacitor (which must be the current flowing) and write:

$$I(t) = \frac{dQ}{dt} = C \times \frac{dV_{max} \exp(j\omega t)}{dt} \tag{20.5}$$

and since V_{max} doesn't change with time, we get:

$$\begin{aligned} I(t) &= C V_{max} \frac{d \exp(j\omega t)}{dt} \\ &= C\, j\omega\, V_{max} \exp(j\omega t) \\ &= C\, j\omega\, V(t) \end{aligned} \tag{20.6}$$

which we could write as:

$$V(t) = I(t) \times \frac{1}{j\omega C} \tag{20.7}$$

This has the same form of Ohm's law (the voltage is proportional to the current); it's just that the resistance has been replaced by an imaginary quantity known as the impedance of the capacitor. We can use this formula to analyse circuits in the same way as Ohm's law for resistors used at DC (but remember, only for sinusoidal voltages and currents: it doesn't work for anything else).

20.4.1 An example of a capacitor circuit

Consider a potential divider consisting of a resistor and a capacitor:

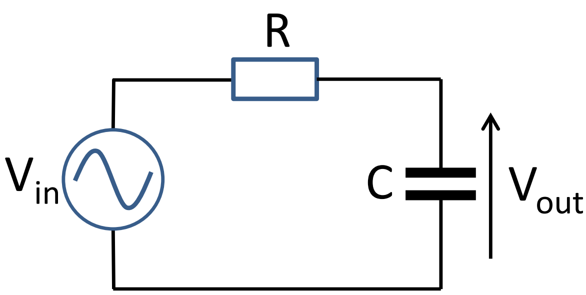

Figure 20.2 Simple RC circuit low-pass filter

[3] See the chapter on "Phasors" if you're unsure why I'm taking one complex sinusoid rather than a real sinusoid here.

The formula for a potential divider (made more general using 'Z' for impedance rather than 'R' for resistance) is:

$$V_{out} = V_{in} \frac{Z_2}{Z_1 + Z_2} \tag{20.8}$$

where Z_2 is the impedance of the component (or network "on the bottom" and Z_1 is the impedance of the component (or network) "on the top". For a resistor, the impedance is just equal to the resistance; for a capacitor the impedance is given by $1 / j\omega C$.

So, in this case, where Z_1 is a resistor and Z_2 is a capacitor, and $\mathbf{V_{out}}$ and $\mathbf{V_{in}}$ are phasors representing the output and input oscillations respectively, we get:

$$\begin{aligned}\mathbf{V_{out}} &= \mathbf{V_{in}} \frac{1/j\omega C}{R + 1/j\omega C} \\ &= \mathbf{V_{in}} \frac{1}{1 + j\omega RC}\end{aligned} \tag{20.9}$$

We can readily convert this to polar form[4]:

$$\mathbf{V_{out}} = \mathbf{V_{in}} \frac{1}{\sqrt{1+\omega^2 R^2 C^2}} \exp\left(j \tan^{-1}\left(-\omega RC\right)\right) \tag{20.10}$$

The relation between the input and output voltages is now complex: it has a magnitude but also a phase. What does that mean?

Well, the magnitude of the phasor $\mathbf{V_{out}}$ tells us the maximum value of the output signal at any point during the sine-wave cycle; in other words it's the zero-to-peak amplitude of the output waveform. The phase tells us the relative phase between the output and the input voltages. If this is not zero, then the maximum value of the output voltage will not occur at the same time as the maximum value of the input voltage. The two sine waves will be offset, as shown in the scope trace and phasor diagram below.

[4] The easiest way is to consider the term $1 + j\omega RC$, and note that this can be expressed as the sum of a real part of one and an imaginary part of $j\omega RC$. Thinking about this point on the Argand diagram, the amplitude of this complex number is the length of the line from the origin to the point $(1, \omega RC)$ which Pythagorus' theorem gives as $\sqrt{(1 + (\omega RC)^2)}$, and the line makes an angle with the real-axis of $\tan^{-1}(\omega RC)$, hence it can be expressed in polar form as $\sqrt{(1 + (\omega RC)^2)}\exp(j \tan^{-1}(\omega RC))$. Then just take the inverse of this expression to give the result required.

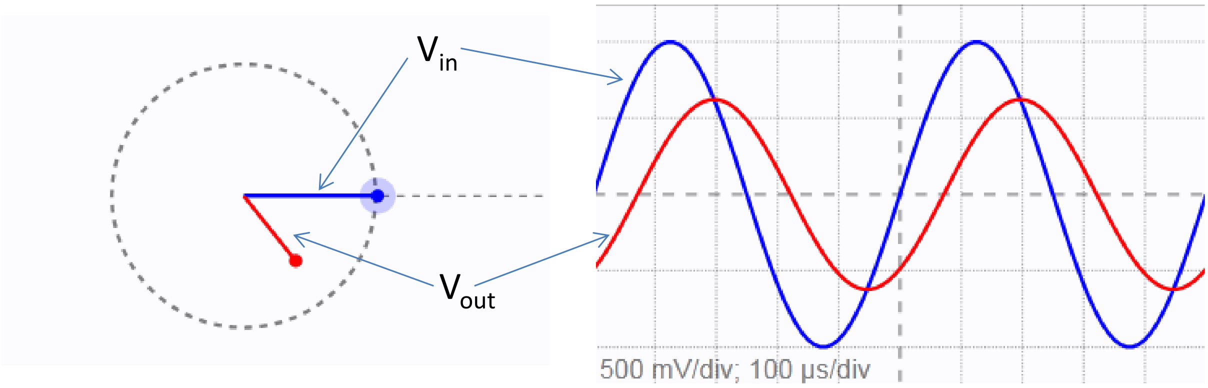

Figure 20.3 Input and output phasors and signals for the low-pass filter

20.4.2 Gains and phases in the capacitor circuit

It's worth taking a moment to consider what equation (20.10) is saying about the behaviour of this circuit at various frequencies.

At DC (when $\omega = 0$) $\mathbf{V}_{\mathbf{out}} = \mathbf{V}_{\mathbf{in}}$ so there is no phase difference and no attenuation, which is reasonable since at zero frequency the impedance of the capacitor is infinite. At an infinite frequency $\mathbf{V}_{\mathbf{out}} = 0$, which again is reasonable, since at an infinite frequency, the impedance of a capacitor is zero so there is never any voltage difference across it.

In-between these frequencies, the attenuation of the circuit steadily increases with frequency, and the phase difference between the output and the input slowly changes from zero degrees to minus ninety degrees (in other words the output is lagging the input).

20.5 Inductors and AC circuits

Once you've understood how to deal with capacitors (and that's not easy), inductors shouldn't present too much of a problem. Inductors obey the equation:

$$V(t) = L \times \frac{dI}{dt} \tag{20.11}$$

The voltage across an inductor is equal to the product of the inductance and the rate of change of the current. Again, taking just one complex sinusoidal frequency, we could write:

$$\begin{aligned} V_{\text{max}} \exp(j\omega t) &= L \times \frac{d\left(I_{\text{max}} \exp(j\omega t)\right)}{dt} \\ &= L \times j\omega\, I_{\text{max}} \exp(j\omega t) \\ V &= I \times j\omega\, L \end{aligned} \tag{20.12}$$

where I_{max} is the maximum current through the inductor, and V_{max} is the maximum voltage across the inductor (and being the maxima, these are not functions of time).

This has the same form as Ohm's law, but this time the impedance of the inductor has to be taken to be $j\omega L$.

20.5.1 Example of an inductor circuit

Consider another potential divider, this time consisting of a resistor and an inductor:

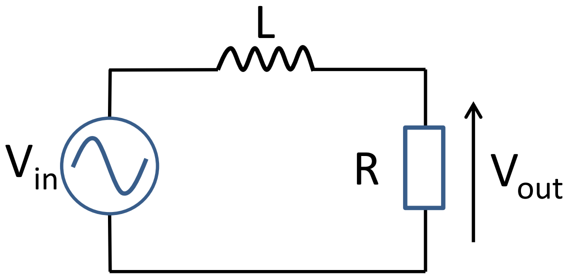

Figure 20.4 Simple low-pass RL filter

Again, for a resistor, the impedance is just equal to the resistance; for an inductor the impedance is taken to be $j\omega L$.

So in this case, where Z_1 is an inductor and Z_2 is a resistor, and again $\mathbf{V}_{out}$ and $\mathbf{V}_{in}$ are phasors representing the output and input oscillations respectively, using the standard potential divider equation gives:

$$\begin{aligned}\mathbf{V}_{out} &= \mathbf{V}_{in}\frac{R}{j\omega L + R} \\ &= \mathbf{V}_{in}\frac{1}{1 + j\omega L / R}\end{aligned} \tag{20.13}$$

Once again, we can convert this to polar form:

$$\mathbf{V}_{out} = \mathbf{V}_{in}\frac{1}{\sqrt{1+\omega^2 L^2 / R^2}}\exp\left(j\tan^{-1}\left(-\omega L / R\right)\right) \tag{20.14}$$

(You might be interested to note that if the components are chosen so that the value of $L/R = RC$, then these last two circuits will behave in exactly the same way. They are both types of low-pass filter (in other words circuits with a larger output at lower frequencies).)

20.6 An example of AC nodal analysis

For a slightly more complex example, consider the circuit below containing a voltage source, two resistors and a capacitor:

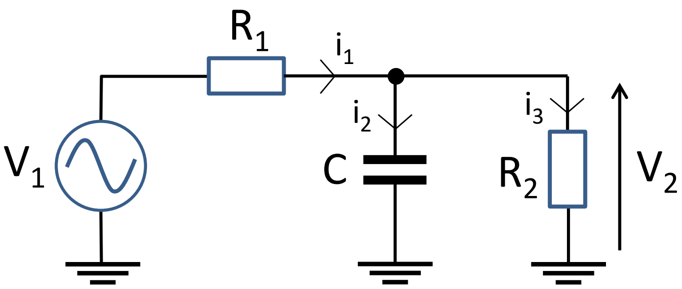

Figure 20.5 A frequency-dependent potential divider

Before analysing any AC circuit we have to decide where the phase reference is going to be. (There is no such thing as absolute phase; the phase of any signal is relative to another signal. We can talk about the “phase difference” between two signals, or the phase of one relative to another, but it makes no sense to say “the phase of this signal is 45 degrees”.) Here, I’ll choose the phase reference to be the phase of the known voltage source V_1.

The simplest way to analyse this circuit is to consider that it is just another potential divider, but this time with the “bottom” impedance set to the parallel combination of the capacitor C and R_2.

All of the techniques of DC analysis work for AC analysis provided the complex impedances of the components are considered, and this includes the formulas for passive linear components in series and parallel. So we can generalise the formula for two resistors in parallel:

$$R = \frac{R_X \times R_Y}{R_X + R_Y} \tag{20.15}$$

to the case of the impedances of any two linear passive components (resistors, capacitors or inductors):

$$Z = \frac{Z_X \times Z_Y}{Z_X + Z_Y} \tag{20.16}$$

In this case, the parallel combination of the resistor R_2 and the capacitor C gives an equivalent impedance of:

$$Z = \frac{R_2 / j\omega C}{R_2 + 1/j\omega C} = \frac{R_2}{1 + j\omega CR_2} \tag{20.17}$$

Writing the phasor representation of the input voltage source as $\mathbf{V_1}$ and the phasor representation of the voltage across the resistor as $\mathbf{V_2}$, then using the potential divider equation gives an output voltage phasor of:

$$\mathbf{V_2} = \mathbf{V_1} \frac{\dfrac{R_2}{1 + j\omega CR_2}}{R_1 + \dfrac{R_2}{1 + j\omega CR_2}} \tag{20.18}$$

which after a small amount of tedious algebra gives:

$$\mathbf{V_2} = \mathbf{V_1} \frac{R_2}{R_1 + R_2 + j\omega CR_1R_2} \tag{20.19}$$

Once $\mathbf{V_2}$ is known, all of the currents can be determined by applying Ohm’s law to the circuit:

$$\mathbf{i_1} = \frac{\mathbf{V_1} - \mathbf{V_2}}{R_1} = \frac{\mathbf{V_1}}{R_1}\left(1 - \frac{R_2}{R_1 + R_2 + j\omega C R_1 R_2}\right)$$
$$= \frac{\mathbf{V_1}}{R_1}\left(\frac{R_1 + R_2 + j\omega C R_1 R_2 - R_2}{R_1 + R_2 + j\omega C R_1 R_2}\right) \tag{20.20}$$
$$= \mathbf{V_1}\frac{1 + j\omega C R_2}{R_1 + R_2 + j\omega C R_1 R_2}$$

$$\mathbf{i_2} = \frac{\mathbf{V_2}}{(1/j\omega C)} = \mathbf{V_2} \times j\omega C = \mathbf{V_1}\frac{j\omega C R_2}{R_1 + R_2 + j\omega C R_1 R_2} \tag{20.21}$$

$$\mathbf{i_3} = \frac{\mathbf{V_2}}{R_2} = \frac{\mathbf{V_1}}{R_1 + R_2 + j\omega C R_1 R_2} \tag{20.22}$$

(Alternatively, you could do a nodal analysis, and first apply Kirchhoff's current law to the junction of the three passive components to reveal that:

$$\mathbf{i_1} = \mathbf{i_2} + \mathbf{i_3} \tag{20.23}$$

and then applying Ohm's law, and the variant of Ohm's law that works for capacitors to R_1, R_2 and C gives:

$$\mathbf{V_1} - \mathbf{V_2} = \mathbf{i_1} R_1 \tag{20.24}$$

$$\mathbf{V_2} = \mathbf{i_2} \times 1/j\omega C \tag{20.25}$$

$$\mathbf{V_2} = \mathbf{i_3} \times R_2 \tag{20.26}$$

That's four equations in four unknowns. From here on it's just a lot of tedious algebra, substituting one equation into another, which would lead to the same answers.)

Some things are immediately apparent: for example the phases of $\mathbf{V_2}$ and all of the currents are non-zero, suggesting that the source voltage $V_1(t)$ and the current supplied by this voltage source are not in phase (except at DC where ω is zero).

If we want the amplitude of these voltages and currents, and their phase relative to the reference voltage, then we have to determine the magnitude and phase of these complex expressions. In many cases, this is most easily done by converting both the numerator and denominator of the relevant complex expression into polar form, and then dividing. For example, to work out $\mathbf{i_2}$, we would note that:

$$j\omega R_2 C = \omega R_2 C \exp\left(j\frac{\pi}{2}\right) \tag{20.27}$$

and using Pythagorus and the argument of the complex number:

$$R_1 + R_2 + j\omega R_1 R_2 C = \sqrt{(R_1+R_2)^2 + (\omega R_1 R_2 C)^2}\exp\left(j\tan^{-1}\left(\frac{\omega R_1 R_2 C}{R_1+R_2}\right)\right) \qquad (20.28)$$

and therefore:

$$\begin{aligned}\frac{j\omega R_2 C}{R_1+R_2+j\omega R_1 R_2 C} &= \frac{\omega R_2 C\exp\left(j\frac{\pi}{2}\right)}{\sqrt{(R_1+R_2)^2+(\omega R_1 R_2 C)^2}\exp\left(j\tan^{-1}\left(\frac{\omega R_1 R_2 C}{R_1+R_2}\right)\right)} \\ &= \frac{\omega R_2 C}{\sqrt{(R_1+R_2)^2+(\omega R_1 R_2 C)^2}}\exp\left(j\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{\omega R_1 R_2 C}{R_1+R_2}\right)\right)\right)\end{aligned} \qquad (20.29)$$

From here, it's easier to see that the magnitude of the current is given by:

$$|\mathbf{i}_2| = V_1\frac{\omega R_2 C}{\sqrt{(R_1+R_2)^2+(\omega R_1 R_2 C)^2}} \qquad (20.30)$$

and the phase of the current phasor $\mathbf{i}_2$ relative to the phase of the voltage phasor $\mathbf{V}_1$ is[5]:

$$\frac{\pi}{2} - \tan^{-1}\left(\frac{\omega R_1 R_2 C}{R_1+R_2}\right) = \tan^{-1}\left(\frac{R_1+R_2}{\omega R_1 R_2 C}\right) \qquad (20.31)$$

It can get much more tedious for more complex expressions; but this is the way to derive expressions for the amplitude and phase for any derived voltage and current in an AC circuit. (No wonder people prefer to use simulators most of the time.)

20.7 A more useful circuit: compensating X10 scope probes

A lot of the circuits that we analyse using phasors are designed to have some sort of filtering action: different frequencies have different gains. One rather interesting exception to this rule is the X10 scope probe, which might not at first seem like a circuit that requires careful AC analysis, but in fact it relies on a clever use of a small capacitor to maintain the accuracy of the scope's readings.

Ideally a scope probe should measure the voltage at a point in a circuit without changing the operation of the circuit being measured. X10 scope probes increase the input impedance of the oscilloscopes by a factor of 10 to reduce the loading effect of the probe on the circuit. However this gives them a problem…

Consider what would happen if you just added a 9 M resistor to a scope probe whose input looked like a 1 M resistor in parallel with a 20 pF capacitor (typical values for scope inputs) in order to get a

[5] In case you've not seen this done before: the result $\pi/2 - \tan^{-1}(x) = \tan^{-1}(1/x)$ comes from considering a right angled triangle: if one (non right-angle) angle has a tangent of x (so it's $\tan^{-1}(x)$ degrees), then the other (non right-angle) angle must be $\pi/2$ minus the first one, and have a tangent of $1/x$. So $\tan(\pi/2 - \tan^{-1}(x)) = 1/x$, and the result follows. It's often a useful result when dealing with complex numbers.

total input resistance of 10 M (ten times greater than the 1 M of the scope alone). You'd get something like the following circuit:

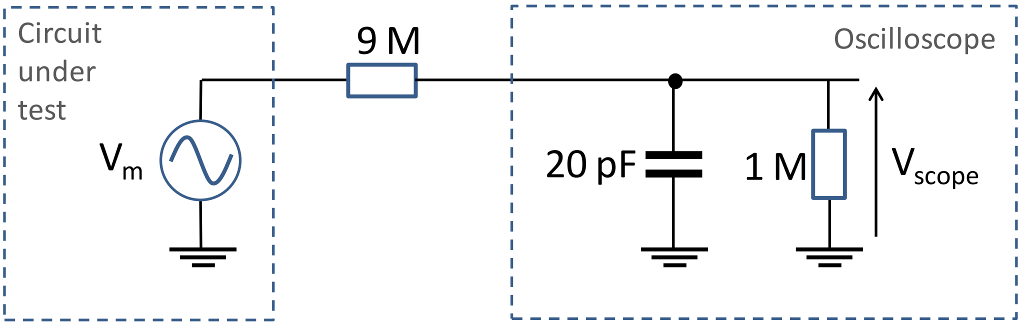

Figure 20.6 An uncompensated X10 scope probe

where $V_{scope}(t)$ is the signal amplified and displayed on the scope's screen, and $V_m(t)$ is the voltage being measured.

This circuit has a definite frequency response: it doesn't treat all frequencies equally. We've already analysed this circuit above; the response is the same as the circuit described in equation (20.19) with R_1 = 9 M, R_2 = 1 M and C = 20 pF. So the frequency response of this circuit is (in terms of the phasor representations of $V_{scope}(t)$ and $V_m(t)$):

$$\mathbf{V}_{\text{scope}} = \mathbf{V}_{\text{m}} \frac{1}{10 + j9\omega C \times 10^6} \tag{20.32}$$

This is not a flat frequency response. Some frequencies will be attenuated more than other ones, which will result in a distortion in the displayed waveform on the oscilloscope.

However, what about the following circuit?

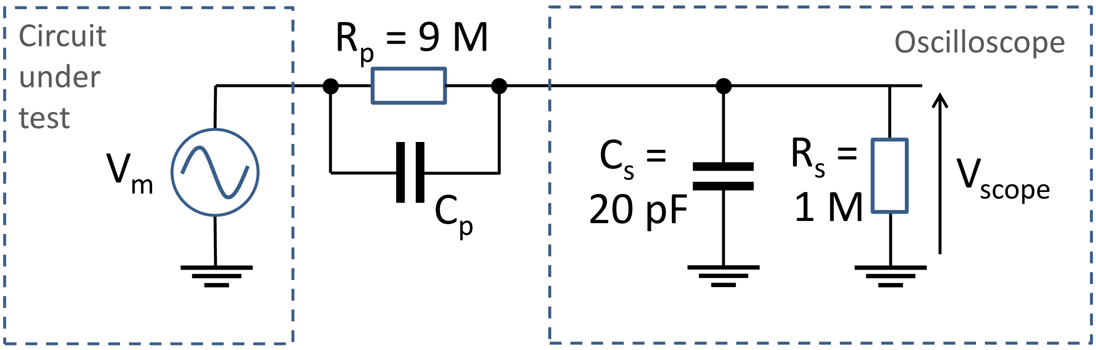

Figure 20.7 A compensated X10 scope probe

This one can similarly be analysed using the potential divider equation, and this time both the "upper" and "lower" impedances are parallel combinations of a resistor and a capacitor. In this case we'd get:

$$\mathbf{V}_{\text{scope}} = \mathbf{V}_{\text{m}} \frac{(R_s / j\omega C_s)/(R_s + 1/j\omega C_s)}{(R_p / j\omega C_p)/(R_p + 1/j\omega C_p) + (R_s / j\omega C_s)/(R_s + 1/j\omega C_s)} \tag{20.33}$$

and after some manipulation of this, we end up with:

$$\mathbf{V}_{\text{scope}} = \mathbf{V}_{\text{m}} \frac{R_s / (1 + j\omega R_s C_s)}{R_p / (1 + j\omega R_p C_p) + R_s / (1 + j\omega R_s C_s)} \tag{20.34}$$

Now R_p is nine times R_s, so we could write:

$$\begin{aligned}
\mathbf{V}_{\text{scope}} &= \mathbf{V}_{\text{m}} \frac{R_s / (1 + j\omega R_s C_s)}{9R_s / (1 + j\omega\, 9R_s C_p) + R_s / (1 + j\omega R_s C_s)} \\
&= \mathbf{V}_{\text{m}} \frac{1}{9 \dfrac{1 + j\omega R_s C_s}{1 + j\omega\, 9R_s C_p} + 1}
\end{aligned} \tag{20.35}$$

Now the clever bit: what happens if you choose C_p (the capacitor across the 9 M resistor) to be one-ninth of the scope's input capacitance C_s? So that $9C_p = C_s$? Then you'd get:

$$\mathbf{V}_{\text{scope}} = \mathbf{V}_{\text{m}} \frac{1}{9 \dfrac{1 + j\omega R_s C_s}{1 + j\omega\, 9R_s C_p} + 1} = \mathbf{V}_{\text{m}} \frac{1}{9 \times \dfrac{1 + j\omega R_s C_s}{1 + j\omega R_s C_s} + 1} = \mathbf{V}_{\text{m}} \frac{1}{9+1} = \mathbf{V}_{\text{m}} \frac{1}{10} \tag{20.36}$$

and that is frequency independent. $\mathbf{V}_{\text{scope}}$ is now just one-tenth of $\mathbf{V}_{\text{m}}$ at all frequencies.

For this to work, the capacitance in the scope probe C_p has to be set accurately to one-ninth of the scope's input capacitance, and that's what compensating a scope probe does: in many real scope probes C_p is a variable capacitor, and can be adjusted to match whatever scope input the probe is being used with. (An alternative solution is to fix the value of C_p, and effectively make C_s a variable capacitor by adding putting a small variable capacitor between the signal and ground.)

(Note that in this analysis I've neglected the capacitance of the cable connecting the probe to the oscilloscope. In practice, this cable has a significant effect on the circuit, however it can be included in the compensation in exactly the same way.)

20.8 Summary: the most important things to know

- All the techniques used for the analysis of linear DC circuits (nodal analysis, Thévenin equivalent circuits, source transformations, potential and current dividers) can all be used for AC circuits as well provided:
    - Complex impedances are used instead of resistances
    - Phasor representations of the voltages and currents are used rather than their constant (DC) values
    - Only sinusoidal inputs and outputs are considered
    - Only linear components are used in the circuits