19 A Short Introduction to Complex Impedance

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Prerequisite knowledge required: Phasors, Ohm's Law and Kirchhoff's Laws

19.1 Introduction

In the chapter on phasors, we saw how using complex numbers to represent sinusoidal oscillations makes adding and subtracting sinusoidal oscillations much easier to do. Applying Kirchhoff's voltage and current laws requires a lot of adding and subtracting of voltages, so this gives us a convenient method for applying Kirchhoff's laws to circuits in which there are sinusoidal voltage and current sources at the same frequency (but different phases).

However you can't solve a circuit just by using Kirchhoff's laws. You need some way to relate the voltage across the components to the current through them. Ohm's law is fine for resistors, but what about capacitors and inductors?

It turns out that the use of complex numbers (as in phasors) provides a way to extend Ohm's law to cover capacitors and inductors. This is the subject of this chapter: the technique of complex impedances. This is another topic which can cause a lot of confusion, and I suspect that this is because there's a trick to the use of complex impedances to analyse AC circuits which is often glossed over, or at least not described very well[1]. This chapter is my attempt to explain what's going on. (If this gets a bit confusing, you can just skip to the summary at the end: knowing how to use complex impedances is the most important thing here.)

19.2 Ohm's law and capacitors

Ohm's law only applies to resistors (by definition: anything that doesn't obey Ohm's Law isn't a resistor[2]). Ohm's law states that the current through a resistor is proportional to the voltage across it. The ratio of the voltage to the current is known as the resistance of the resistor, and it's measured in ohms.

Resistance is a real number, so the ratio of the voltage across the resistor to the current going through it is constant. This means that when the voltage is at a maximum, the current will be too, so the voltage across the resistor and the current through it always have the same phase.

However, consider a capacitor: the current flowing into one side (and out of the other side) of a capacitor is not proportional to the voltage across it. Capacitors obey a completely different law:

$$Q = CV \tag{19.1}$$

where Q is the charge stored on each plate of the capacitor, C is the capacitance and V is the voltage across the capacitor. For a capacitor, it's not the current that's proportional to the voltage, it's the charge.

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[1] At least it wasn't explained well to me when I was trying to learn this stuff.

[2] It's a bit of a strange law, since it only applies to things that obey it. However, since a lot of real physical things do obey Ohm's law (at least approximately) it turns out to be very useful in practice.

Current occurs when charge moves around, so if the charge on the capacitor's plates change, then a current must have flowed onto one plate (and off from the other plate) of the capacitor. In fact, since current is the rate of flow of charge, we can write:

$$I(t) = \frac{dQ(t)}{dt} \tag{19.2}$$

If the current is held constant, the rate of change of charge will be constant, which means that the capacitor will just charge up (or down) at a constant rate. A constant current implies a steadily increasing (or decreasing) voltage.

Put these two equations together, and we get:

$$I(t) = \frac{dQ(t)}{dt} = C\frac{dV(t)}{dt} \tag{19.3}$$

Obviously we can't use Ohm's Law (and all the corresponding circuit analysis equations such as the potential divider equation, or the equations for series and parallel resistors) on capacitors, since the current is not proportional to the voltage. Capacitors are not resistors: they don't obey Ohm's law.

19.2.1 Oscillating signals

Clearly capacitors don't obey Ohm's law in general, but what about the special case we're interested in here? A single frequency input? Suppose that the signal in the circuit was a sinusoidal oscillation, something that could be represented by:

$$V(t) = \cos(\omega t) \tag{19.4}$$

Then we could write:

$$\begin{aligned} \frac{dV(t)}{dt} &= -\omega \sin(\omega t) \\ &= -\omega \sqrt{1-\cos^2(\omega t)} \\ &= -\omega \sqrt{1-V^2(t)} \end{aligned} \tag{19.5}$$

and that gives us:

$$I(t) = C\frac{dV(t)}{dt} = -C\omega \sqrt{1-V^2(t)} \tag{19.6}$$

Well, it's a little closer to what we'd like (the current proportional to the voltage). We've got rid of the differentiation at least. But it's not quite there yet. We still can't apply Ohm's law to this capacitor.

19.2.2 However…

Suppose that the signal we were putting into the circuit wasn't a real cosine, but was in fact a complex cisoidal oscillation with amplitude A and phase θ, for example:

$$x(t) = A\exp\left(j(\omega t + \theta)\right) \tag{19.7}$$

Then going through the process above reveals that:

$$\begin{aligned} \frac{dV(t)}{dt} &= j\omega \exp\left(j(\omega t + \theta)\right) \\ &= j\omega V(t) \end{aligned} \tag{19.8}$$

and that gives us:

$$\begin{aligned} I(t) &= C\frac{dV(t)}{dt} = j\omega C V(t) \\ V(t) &= I(t)\frac{1}{j\omega C} \end{aligned} \tag{19.9}$$

Ah! Now we have something that seems to obey Ohm's law. The current is proportional to the voltage. For this strange complex oscillating signal, the capacitor appears to be behaving just like a resistor with an impedance of 1 / *jωC*.

19.2.3 But the signal we're interested in isn't a complex cisoid, it's a real cosine…

Good point. However, there's another way to write a cosine. If you consider Euler's famous equation:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{19.10}$$

It's quite straightforward[3] to prove that:

$$\cos(\theta) = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \tag{19.11}$$

And hence for an oscillation with amplitude *A* and relative phase θ:

$$A\cos(\omega t + \theta) = \frac{A}{2}\left(e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right) \tag{19.12}$$

In other words, a cosine can be expressed as the sum of two complex cisoidal oscillations, one with a positive frequency and one with a negative frequency, each with an amplitude of one-half of the original real signal. And we've just proved that for a complex cisoidal oscillation, capacitors behave just like resistors, although with an impedance of 1 / *jωC*.

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[3] Noting that cosine is an even function so cos(-θ) = cos(θ) and sine is an odd function so sin(-θ) = -sin(θ), we get:

$$\begin{aligned} \exp(j\theta) + \exp(-j\theta) &= \cos(\theta) + j\sin(\theta) + \cos(-\theta) + j\sin(-\theta) \\ &= \cos(\theta) + j\sin(\theta) + \cos(\theta) - j\sin(\theta) \\ &= 2\cos(\theta) \end{aligned}$$

Therefore, to a cosine wave, capacitors should behave just like resistors with an impedance of $1/j\omega C$… except we just proved above that they don't. Oh dear, what's going on?

19.2.4 The trick (you might not like this)…

The problem is that we've just expressed the cosine in terms of two cisoidal oscillations, one with a positive frequency (ω) and one with a negative frequency ($-\omega$). This means that for the positive frequency, the capacitor behaves like a resistor with an impedance of $1 / j\omega C$, and for the negative frequency, it behaves like a resistor with an impedance of $1 / j(-\omega)C = -1 / j\omega C$. So the capacitor doesn't behave the same way for the two components of the cosine wave.

One way round this problem would be to use superposition: first do an analysis of the circuit for positive frequencies using a capacitor impedance of $1 / j\omega C$, and then do another analysis of the circuit for negative frequencies using a capacitor impedance of $-1 / j\omega C$, and then add up the two results. Provided the circuit is linear[4] that would work, but it's rather long-winded. Fortunately there's a short-cut.

How about if we just consider the positive frequency component? Just don't worry about the negative frequency component for now? That would give as an answer (for the voltage, or current, or whatever we're trying to work out) a complex cisoidal oscillation with a certain amplitude and phase. If we call this component of the answer $y_1(t)$, we could write something like:

$$\begin{aligned} y_1(t) &= B\exp\left(j(\omega t+\phi)\right) \\ &= B\cos(\omega t+\phi) + jB\sin(\omega t+\phi) \end{aligned} \tag{19.13}$$

The trick is to use the fact that whatever we're trying to work out (the actual voltage in the circuit, or the actual current through a component), the answer must be real. It can't have any imaginary component (there's no such thing as an imaginary voltage in real life). So whatever happens to the negative frequency component, the result of adding it back to produce the final answer must cancel out the imaginary component of equation (19.13). Since we also know that the result is going to be an oscillation with a frequency of $-\omega$, there's really only one thing this negative frequency component can be: the complex conjugate of the solution for the positive frequency[5].

$$\begin{aligned} y_2(t) &= B\exp\left(j(-\omega t-\phi)\right) \\ &= B\cos(-\omega t-\phi) + jB\sin(-\omega t-\phi) \\ &= B\cos(\omega t+\phi) - jB\sin(\omega t+\phi) \end{aligned} \tag{19.14}$$

(using the fact that cos(-x) = cos(x) and sin(-x) = -sin(x)).

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[4] And throughout this whole analysis we've been assuming that the circuit is component (double the input and the output doubles). If it isn't then we couldn't use Ohm's law anyway, so this isn't a problem in practice.

[5] There's another way of thinking about this that you might find more convincing. If you write out all the circuit equations using the negative frequency, you'll find that you have the complex conjugate of the circuit equations for the positive frequency case. Therefore the result of solving them will be the complex conjugate of the positive frequency case. So there's no need to solve them: just take the positive frequency case that's already been solved, and add the complex conjugate of that result to get the actual real voltages and currents.

Adding these two together then results in:

$$\begin{aligned} v(t) &= y_1(t) + y_2(t) \\ &= B\cos(\omega t + \phi) + jB\sin(\omega t + \phi) + B\cos(\omega t + \phi) - jB\sin(\omega t + \phi) \\ &= 2B\cos(\omega t + \phi) \end{aligned} \tag{19.15}$$

which is entirely real (as any actual voltage or current measured in the lab must be).

In effect we've used the fact that we know the final answer must be real to save time working out what happens for negative frequencies. This allows us to just consider the positive frequency component, and do one calculation, just like Ohm's law allows us to do at DC. Having done the calculation, we just throw away the imaginary part of the answer (since we know this will be cancelled out by the imaginary part of the answer for the negative frequency component) and double the real part (since the real part of the negative frequency component is the same as the real part of the positive frequency component, the real part of the resultant sum will be twice the real part of the positive frequency component $y_1(t)$).

19.2.5 Why bother to halve the amplitude at the start and then double it at the end?

No reason at all, really. Strictly speaking it should be done since that's where the maths comes from, but in practice it's a bit of a waste of time. We could dispense with both operations, and this is standard practice. This final simplification completes our technique, which could be summarised as:

1. Convert all sinusoidal signals into cisoidal signals with a real part equal to the original sinusoidal signal[6].
2. Analyse the circuit using the familiar techniques (Ohm and Kirchhoff's laws), but with the impedance of the capacitors set to $1 / j\omega C$.
3. After the analysis, take the real part of the answer.

That's it. It works for inductors as well (and any combination of inductors, resistors and capacitors), but for inductors the impedance must be set[7] to $j\omega L$.

19.2.6 Complex impedance and phasors

You might notice that the first step in the method described above is almost the same thing as finding the phasor representation of the signals. If the signals are already described in phasor form, then the first step just requires multiplying by $\exp(j\omega t)$.

However, it's not worth doing this. Since this factor of $\exp(j\omega t)$ appears in every voltage and every current, we can immediately cancel it out from any application of Ohm or Kirchhoff's laws to the circuit just by dividing all terms in the equations by $\exp(j\omega t)$. So we usually don't even bother writing it down. This means we are just using the phasor representations of the signals in these

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[6] Equal to, and not half of, since we're going to dispense with the factor of two in this method. Quick reminder: strictly speaking the positive frequency component of $\cos(\omega t)$ is $1/2 \exp(j\omega t)$.

[7] I'll leave this one as an exercise to the reader. For an inductor of L Henrys, the relevant equation is that the voltage is equal to L times the rate of change of current: $V(t) = L\dfrac{dI(t)}{dt}$.

equations. The factor of exp($j\omega t$) can be added in at the end of the calculations, just before taking the real part to reveal the actual observed signal.

So now the only difference between the calculations in AC circuit analysis and those for DC circuit analysis is that in AC we're using complex numbers (the phasor representations of the voltages and currents) rather than real numbers (the actual voltages and currents), and we're using complex impedances for capacitors and inductors.

For example, consider the circuit below, where a sinusoidal voltage source is driving a capacitor through a resistor:

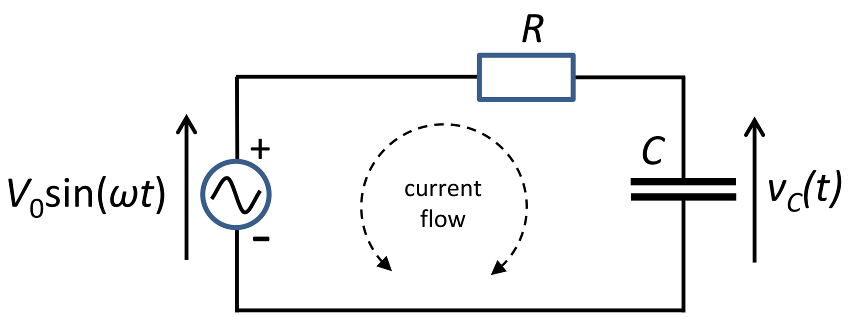

Figure 19.1 An AC potential divider

The first step is to express the voltage source in phasor form. Since it's a sine wave (rather than a cosine wave) this means that the required phasor is $-jV_0$ (with an argument of $-\pi/2$) since:

$$\begin{aligned}\Re\left\{-jV_0\exp(j\omega t)\right\} &= \Re\left\{-jV_0\left(\cos(\omega t)+j\sin(\omega t)\right)\right\}\\ &= \Re\left\{-jV_0\cos(\omega t)-j^2V_0\sin(\omega t)\right\}\\ &= V_0\sin(\omega t)\end{aligned} \tag{19.16}$$

Then, using the complex impedance of the capacitor, the standard potential divider equation can be used to compute the phasor representation of the voltage across the capacitor:

$$\begin{aligned}\mathbf{v_c} &= -jV_0\frac{1/j\omega C}{R+1/j\omega C}\\ &= -jV_0\frac{1}{1+j\omega RC}\end{aligned} \tag{19.17}$$

The voltage across the capacitor can then be determined by multiplying this by exp($j\omega t$) and taking the real part:

$$v_c(t) = \Re\left\{\mathbf{v_c}\exp(j\omega t)\right\} = \Re\left\{-jV_0\frac{1}{1+j\omega RC}\exp(j\omega t)\right\} \tag{19.18}$$

However this calculation is easier to do if the phasor representation is first converted into polar form:

$$\frac{-jV_0}{1+j\omega RC}=\frac{V_0\exp(-j\pi/2)}{\sqrt{1+\omega^2R^2C^2}\exp\left(j\tan^{-1}(\omega RC)\right)}$$
$$=\frac{V_0}{\sqrt{1+\omega^2R^2C^2}}\exp\left(-j\left(\frac{\pi}{2}+\tan^{-1}(\omega RC)\right)\right) \tag{19.19}$$

and knowing how phasors are related to the real signals, we can immediately identify the amplitude and phase of the signal, and write the voltage across the capacitor as:

$$v_c(t)=\frac{V_0}{\sqrt{1+\omega^2R^2C^2}}\cos\left(\omega t-\frac{\pi}{2}-\tan^{-1}(\omega RC)\right) \tag{19.20}$$

which is equal to:

$$v_c(t)=\frac{V_0}{\sqrt{1+\omega^2R^2C^2}}\sin\left(\omega t-\tan^{-1}(\omega RC)\right) \tag{19.21}$$

(This is the same problem done in the chapter on time constants and time variant signals using differential equations; it takes rather longer to get to this result that way.)

19.1 Summary: the most important things to know

- When analysing the response of a linear circuit to sinusoidal inputs, you can use the familiar Ohm's law and Kirchhoff's laws to analyse the circuit, using a value of $1 / j\omega C$ as the impedance of the capacitors, $j\omega L$ as the impedance of the inductors, and a complex cisoidal oscillation $A \exp(jwt + \theta)$ to represent a cosine wave signal with amplitude A and phase θ.

  Then apply all the usual analysis, remembering to use the phasor technique of using complex numbers to represent all the voltages and currents. To extract the real voltage or current at any point, just multiply the phasor by $\exp(j\omega t)$ and take the real part of the result.

- This technique can only be used for linear circuits.
- If the input is not at a single-frequency, its necessary to express the input in terms of the sum of a number of single frequencies, apply this technique to each component frequency, and then add up all the results for each component.
  - This is a very common and popular thing to do… but that's another story for another time.