18 A Short Introduction to Phasors

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Prerequisite knowledge required: Kirchhoff's Laws, Time-Variance and Time Constants, Complex Numbers

18.1 Introduction

The first thing to say is that phasors are really useful in analysing AC circuits, and it's well worth spending the time required to get your head around them. (I thought you might need a bit of encouragement, as for those not very familiar with complex numbers, they can be a bit confusing at first.)

On with the story: a phasor[1] (short for "phase vector") is a two-dimensional vector that represents a sinusoidal oscillation: anything that can be written in the form:

$$x(t) = A\cos(\omega t + \theta) \tag{18.1}$$

The obvious question to ask at this point would be "why would anyone want to represent an oscillating signal like this as a two-dimensional vector?".

The rather unhelpful answer is often that they don't. Well, not directly anyway. It's a side-effect of something that is very useful, and that's to represent oscillations as complex numbers. Since you can represent a complex number in terms of a line from the origin to the complex number in the Argand diagram, and that line has a direction and a magnitude, it can be thought of as a vector, and that's a convenient way to draw things on diagrams. It's a two-dimensional vector, since it is restricted to lie in the two-dimensional plane of the Argand diagram; but for our purposes the fact that it's a complex number is more important.

Of course all this answer does is immediately prompt the question "why would anyone want to represent an oscillating signal as a complex number?". That's an easier one to answer: because it makes the maths of working out what happens in a circuit much, much easier. It turns out that if we do this, we can extend Ohm's law and Kirchhoff's laws to cover capacitors and inductors, and then apply these laws to AC circuits in exactly the same way as we analyse DC circuits. How this is done is the subject of the next chapter; for now it's enough to just get used to phasors.

18.2 Adding voltages and currents

The beauty of the phasors approach is that it makes adding up sinusoidal waveforms much easier to do, and when applying Kirchhoff's voltage and current laws in circuits which operate at a specific frequency, this is something that we have to do a lot.

For example, consider a circuit which has two voltage sources at the same frequency, but different phases, connected by a resistor:

[1] Not to be confused with a phaser (which is what Star Trek people shoot each other with).

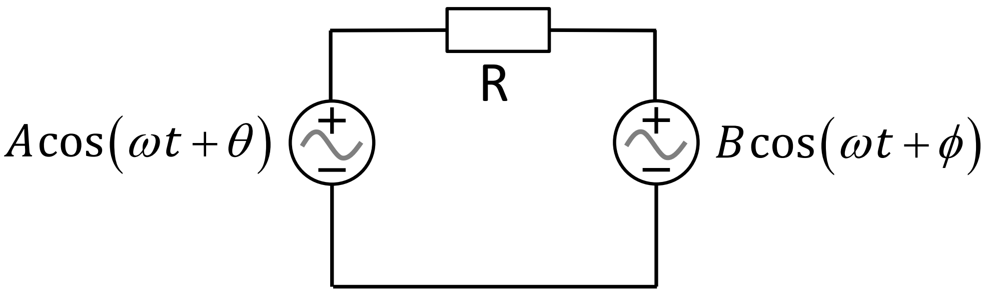

Figure 18.1 Kirchhoff's voltage law in an AC circuit

If we wanted to know the voltage across the resistor, we could just write it as:

$$V_R(t) = A\cos(\omega t + \theta) - B\cos(\omega t + \phi) \tag{18.2}$$

but it would be much neater and more convenient to express this voltage in the form:

$$V_R(t) = C\cos(\omega t + \varphi) \tag{18.3}$$

and this is always possible, since adding up (or subtracting) two cosine waves at the same frequency results in another cosine wave at the same frequency[2]. The only problem is how to work out C and ψ from knowledge of A, B, θ and φ.

18.2.1 Adding and subtracting sinusoidal voltages: the slow way

This sum can be done just using trigonometric identities. The idea is that you first assume that a solution exists, and then set out to find it. Assuming a solution exists, then there must be some value of C and ψ for which:

$$C\cos(\omega t + \varphi) = A\cos(\omega t + \theta) - B\cos(\omega t + \phi) \tag{18.4}$$

for all values of the time variable t. Using the formula:

$$\cos(X + Y) = \cos(X)\cos(Y) - \sin(X)\sin(Y) \tag{18.5}$$

equation (18.4) can be expanded into the form:

$$\begin{aligned} C\cos(\omega t)\cos(\varphi) - C\sin(\omega t)\sin(\varphi) \\ = A\cos(\omega t)\cos(\theta) - A\sin(\omega t)\sin(\theta) \\ -B\cos(\omega t)\cos(\phi) + B\sin(\omega t)\sin(\phi) \end{aligned} \tag{18.6}$$

This must be true for all times, so first consider time $t = 0$, when $\cos(\omega t) = 1$ and $\sin(\omega t) = 0$:

$$C\cos(\varphi) = A\cos(\theta) - B\cos(\phi) \tag{18.7}$$

[2] This isn't obvious, but it can be proved by first assuming it's true, and then finding values of C and ϕ for which equation (18.4) is satisfied at all times.

then consider time $t = \pi/2\omega$, when $\cos(\omega t) = 0$ and $\sin(\omega t) = 1$:

$$-C\sin(\varphi) = -A\sin(\theta) + B\sin(\phi) \tag{18.8}$$

Divide equation (18.8) by equation (18.7) and multiplying by minus one, and we get:

$$\tan(\varphi) = \frac{A\sin(\theta) - B\sin(\phi)}{A\cos(\theta) - B\cos(\phi)} \tag{18.9}$$

from which $\tan(\psi)$ and then ψ can be determined[3].

Working out C is slightly simpler, since squaring both equations (18.8) and (18.7), and then adding the results gives:

$$\begin{aligned} C^2\left(\sin^2(\phi) + \cos^2(\phi)\right) &= \left(A\cos(\theta) - B\cos(\phi)\right)^2 + \left(-A\sin(\theta) + B\sin(\phi)\right)^2 \\ C^2 &= A^2 + B^2 - 2AB\left(\cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi)\right) \\ C &= \sqrt{A^2 + B^2 - 2AB\cos(\theta - \phi)} \end{aligned} \tag{18.10}$$

and since C is an amplitude, and therefore always positive, there is no confusion about which square root is required: the positive one is the right one.

(You might notice that this is exactly the same equation as the cosine rule, which gives the length C of a triangle with sides of length A, B, and C, where the sides of length A and B are separated by an angle of $(\theta - \varphi)$. This is not a coincidence; for why this happens, please read on.)

While this approach to finding the resultant of adding two sine waves works fine, it is rather time-consuming, and there is that tricky four-quadrant inverse tangent. It would be nice if there was a faster way to do the calculation. Fortunately, there is, but it requires the use of complex numbers. It goes like this:

18.2.2 Adding and subtracting sinusoidal voltages: the faster way

First, notice that from Euler's equation:

$$\exp(jx) = \cos(x) + j\sin(x) \tag{18.11}$$

we can express any sinusoidal oscillation as the real part of a complex cisoidal oscillation. For example, here:

$$\begin{aligned} A\cos(\omega t + \theta) &= \Re\left\{A\cos(\omega t + \theta) + jA\sin(\omega t + \theta)\right\} \\ &= \Re\left\{A\exp\left(j(\omega t + \theta)\right)\right\} \end{aligned} \tag{18.12}$$

[3] Although you have to be careful here, since tan(X) = tan(180 + X), so just knowing the tangent of the angle doesn't uniquely identify the angle. You really need to do a four-quadrant inverse tangent to ensure you get the right angle.

and:

$$\begin{aligned} B\cos(\omega t+\phi) &= \Re\{B\cos(\omega t+\phi)+jB\sin(\omega t+\phi)\} \\ &= \Re\{B\exp(j(\omega t+\phi))\} \end{aligned} \tag{18.13}$$

It's worth pausing at this point to examine these complex numbers a bit more closely. The first complex number can be written as:

$$A\exp(j(\omega t+\theta)) = A\exp(j\theta)\times\exp(j\omega t) \tag{18.14}$$

by expanding out the exponential term. This shows that this is the product of a complex number with the magnitude equal to the zero-to-peak amplitude of the original real sinusoidal signal and an argument equal to the phase of the original signal at time t = 0, and a unit cisoidal oscillation exp($j\omega t$).

A similar conversion can be done for the other real sinusoidal voltage source:

$$B\exp(j(\omega t+\phi)) = B\exp(j\phi)\times\exp(j\omega t) \tag{18.15}$$

Now if the voltage across the resistor is the difference between the real parts of two complex numbers, then it must be the real part of the difference between the two complex numbers[4], and the difference between the two complex numbers in equations (18.14) and (18.15) is:

$$=(A\exp(j\theta)-B\exp(j\phi))\exp(j\omega t) \tag{18.16}$$

The voltage across the resistor must be the real part of this, which is:

$$C\cos(\omega t+\varphi) = \Re\{(A\exp(j\theta)-B\exp(j\phi))\exp(j\omega t)\} \tag{18.17}$$

Just like we expressed the two sinusoidal voltage sources as the real part of a complex number, we can express the voltage across the resistor as the real part of a complex number as well. In this case, we'd get:

$$C\cos(\omega t+\varphi) = \Re\{C\exp(j\varphi)\exp(j\omega t)\} \tag{18.18}$$

and again, the term Cexp($j\psi$) has a magnitude equal to the zero-to-peak amplitude of the oscillation, and an argument equal to the phase of the oscillation at time t = 0.

Comparing equations (18.17) and (18.18), it's hopefully obvious that:

$$C\exp(j\varphi) = A\exp(j\theta)-B\exp(j\phi) \tag{18.19}$$

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[4] If this isn't obvious, consider two complex numbers A + jB and C + jD. The real parts are A and C, so the difference between the real parts is A – C. Doing the subtraction before taking the real part gives the same thing, since the difference between the two complex numbers is (A – jB) + (C – jD) = (A – C) + j(B – D), and the real part of this is (A – C).

In other words: to work out the difference between two sinusoidal oscillations at the same frequency, we can:

1. Produce the complex number with a magnitude equal to the zero-to-peak amplitude of the first oscillation, and an argument equal to the phase of the first oscillation at time $t = 0$.
2. Do the same thing for the second oscillation.
3. Subtract the second complex number from the first complex number.
4. The result is a complex number which has an amplitude equal to the magnitude of the resultant oscillation, and an argument equal to the phase of the resultant oscillation at time $t = 0$.

A similar process works for adding sinusoidal oscillations of the same frequency as well; it just requires an addition at stage three rather than the subtraction.

18.3 Introduction to phasors

The complex numbers that we used in the last section are phasors. (A more formal definition of a phasor is the complex number which when multiplied by exp($j\omega t$), and then the real part taken of the result, gives the signal represented by the phasor.)

From now on, I'll write phasors in bold (like this: **z**), so it's clearer when I'm talking about the phasor, and when I'm referring to the actual time-domain oscillation that it represents (which I'll write as $z(t)$).

The result can be derived in a slightly different way: if phasor **z** represents the single-frequency oscillation given in equation (18.1), then z must satisfy:

$$A\cos(\omega t+\theta)=\Re\{\mathbf{z}\exp(j\omega t)\} \tag{18.20}$$

Writing the complex number **z** in polar form:

$$\mathbf{z}=r\exp(j\phi) \tag{18.21}$$

and applying Euler's result to this reveals that:

$$\begin{aligned}A\cos(\omega t+\theta)&=\Re\{r\exp(j\phi)\exp(j\omega t)\}\\&=\Re\{r\exp(j(\omega t+\phi))\}\\&=\Re\{r\cos(\omega t+\phi)+j\sin(\omega t+\phi)\}\\&=r\cos(\omega t+\phi)\end{aligned} \tag{18.22}$$

from which it's obvious that $A = r$ and $\theta = \varphi$. In other words, the phasor that represents a complex oscillation with zero-to-peak amplitude of A and a phase of θ at time $t = 0$ is the complex number with magnitude[5] A and phase angle θ.

For example, consider an oscillation with a zero-to-peak amplitude of 1.5 volts, and a phase of 135 degrees relative to a given reference phase. This could be expressed as:

$$x(t) = 1.5\cos\left(\omega t + \frac{3\pi}{4}\right) \tag{18.23}$$

In terms of a phasor, this would be a vector with magnitude 1.5, and a direction 135 degrees round anti-clockwise from a reference phase (which conventionally points right), since this corresponds to the complex number -1.06 + 1.06j, which when written in exponential form is:

$$-1.06 + 1.06j = 1.5\exp\left(j\frac{3\pi}{4}\right) \tag{18.24}$$

and

$$\begin{aligned} \Re\left\{1.5\exp\left(j\frac{3\pi}{4}\right)\exp\left(j\omega t\right)\right\} &= \Re\left\{1.5\exp\left(j\left(\omega t + \frac{3\pi}{4}\right)\right)\right\} \\ &= 1.5\cos\left(\omega t + \frac{3\pi}{4}\right) \end{aligned} \tag{18.25}$$

This can be illustrated as shown in Figure 18.2.

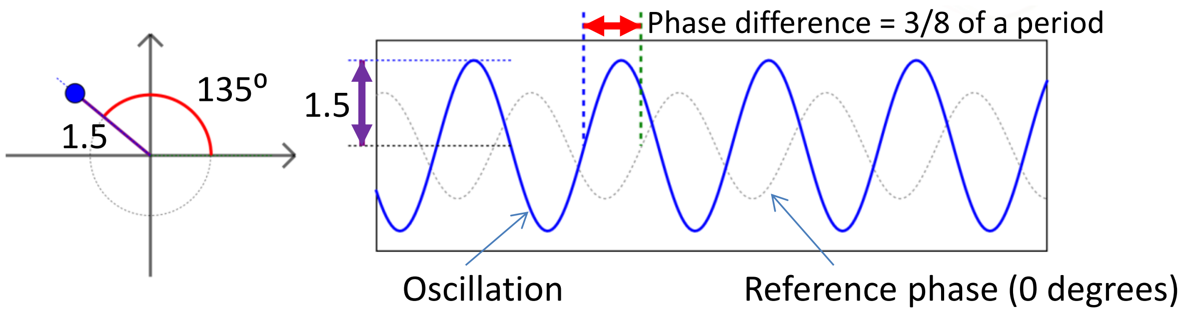

Figure 18.2 Illustration of a phasor and the oscillation it represents

It's worth repeating that the phasor representation of a sinusoidal oscillation doesn't tell you anything about the frequency of the oscillation, only its amplitude and phase. The time domain representation tells you the amplitude, the phase and the frequency.

[5] I should mention at this point that some authors consider "rms phasors" as well as these "zero-to-peak" phasors. An "rms phasor" has a magnitude equal to the rms amplitude of the sinusoidal signal, not the zero-to-peak amplitude. I'll only use "zero-to-peak" phasors (unless specifically noted otherwise).

18.4 Phasor diagrams

Finally, we can get to phasor diagrams. These are diagrams showing the phasors representing the complex numbers used to represent the voltages and currents in a circuit. They can be very useful, since they show at a glance the phase differences between different voltages in the circuit, and between the voltages and currents in the circuit.

For example, consider the following phasor diagram (shown together with the sinusoids that the phasors represent). It's perhaps easier to see from the phasor diagram that adding together the blue and the red oscillations will produce the larger green one.

It's arguably also easier to spot that the smallest red oscillation is leading the largest green one in phase by around 60 degrees. What is uncontroversial is that the phasor diagram is much easier to draw than the sinusoids that you would see on an oscilloscope screen, and (apart from the frequency) the phasor diagram contains all the same information.

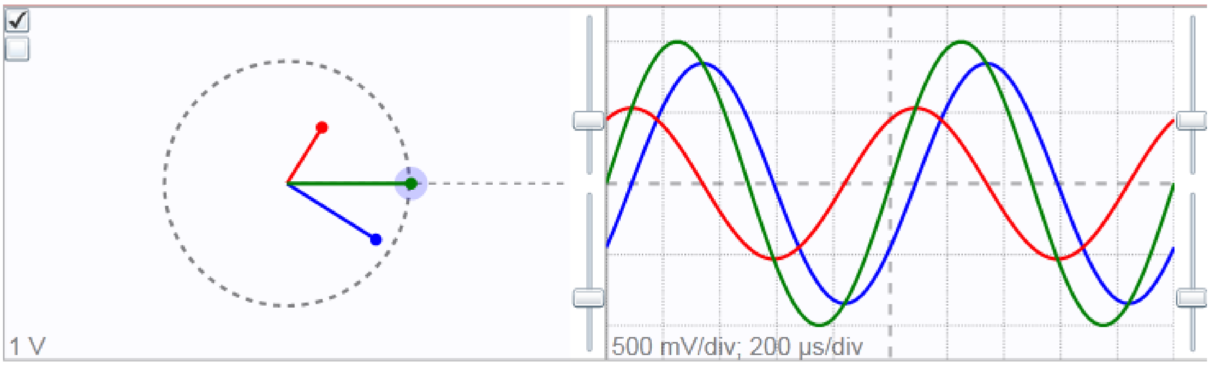

Figure 18.3 Phasors illustrating Kirchhoff's voltage law for an RC circuit

18.5 Summary: the most important things to know

- A phasor is a complex number which represents a real sinusoidal oscillation.
- The magnitude of the phasor is the zero-to-peak amplitude of the real oscillation.
- The argument (phase) of the phasor is the phase of the oscillation at time $t = 0$.
- Phasors do not contain any information about the frequency of the oscillation.
- You can add two oscillations at the same frequency by adding the two phasor representations of the oscillations and converting the result back to the time domain.