6 A Short Introduction to Thévenin and Norton's Theorems

v1.3 – June 2021

Prerequisite knowledge required: Ohm's Law and Kirchhoff's Laws, Resistor Networks

6.1 Introduction

As noted in a previous chapter, Ohm's law:

$$V = I \times R \tag{6.1}$$

and Kirchhoff's laws:

$$\sum_{node} i_n = 0 \qquad \sum_{loop} v_n = 0 \tag{6.2}$$

will allow you to work out the voltages and currents everywhere in most of the circuits we'll be considering in this module[1].

However, for anything other than circuits containing a handful of components, the maths can get very tiresome, as you end up with a lot of simultaneous equations in a lot of unknowns. There are a few short-cuts that can help with the calculations, and one of the most useful of these is the use of the concept of an *equivalent network*: replacing parts of the circuit with a simpler network that behaves in exactly the same way.

These equivalent networks, how to produce them and how to use them, are the subject of this chapter.

6.2 Equivalent networks

Two networks are equivalent if they behave in exactly the same way to any voltage or current applied to their terminals. We've already seen an example of an equivalent network[2] when we looked at resistor networks:

[1] I should include a reminder here that this only works for small, low-frequency circuits where we can assume that Kirchhoff's laws work (see the introduction to Kirchhoff's laws for more details). Also, for circuits with non-linear elements such as diodes, we'll need another equation which relates the voltage across a diode to the current flowing through it.

[2] You might find some people talk about "equivalent circuits" in this context. Engineers can be a bit lazy here and refer to any group of connected components as a "circuit" whether or not there is a loop around which current can flow. I'll try to stick to using "circuit" to refer to something around which current can flow, and "network" for any group of connected components. So all circuits are networks, but not all networks are circuits (for example these resistors are a network, but not a circuit).

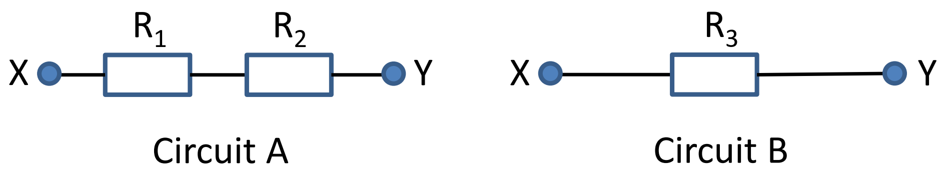

Figure 6.1 Simplest possible example of an equivalent network

Provided that $R_3 = R_1 + R_2$, these resistor networks would behave in exactly the same way to anything connected between the two terminals X and Y.

Rather than using Ohm and Kirchhoff's laws to analyse large circuits, an alternative is to first divide the circuit into a few (often just two) networks, and then find simple equivalent networks for these networks. Then recombine the equivalent networks, and solve the much simpler circuit. This is the essence of Thévenin circuit analysis.

It turns out that for a certain type of very common and useful network (two-terminal linear networks), there is a simple way to find very simple equivalent networks. This is the subject of Thévenin and Norton's theorems.

6.3 Thévenin's theorem

Thévenin's theorem states that for any two-terminal network containing only linear components[3] and fixed voltage and current sources, there always exists an equivalent network that contains just one fixed voltage source, and one resistor, wired in series.

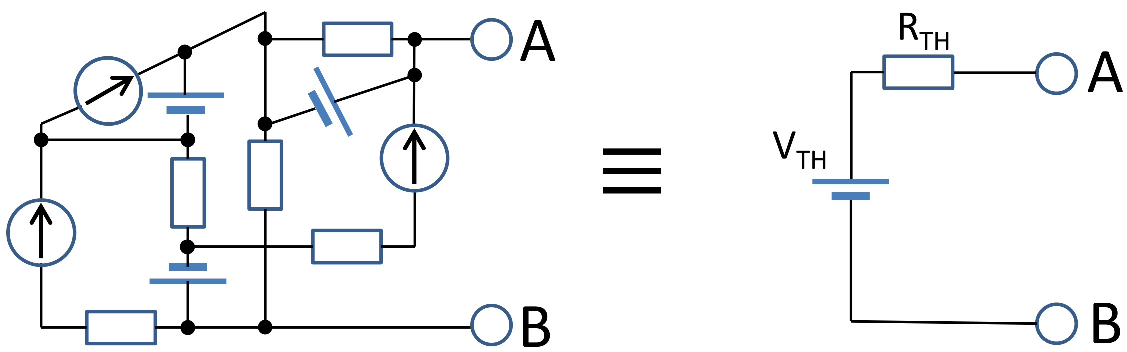

Figure 6.2 - Real and Thévenin equivalent networks. (With the right choice of V_{TH} and R_{TH}, both of these networks would behave in the same way when the terminals A and B are connected to any other network to form a circuit.)

6.3.1 Proving Thévenin's theorem

Knowing how to prove Thévenin's theorem is outside the syllabus of the module, but it's instructive to know how it can be done. First we need to carefully define what we're trying to prove: for any two-terminal linear network, it is always possible to find a value V_{TH} and R_{TH} such that the same

[3] Linear components include resistors, capacitors, inductors, transformers, and linearly-dependent voltage sources and current sources (for example a voltage source which produces a voltage ten times greater than the voltage between two nodes somewhere else in the network). It does not include diodes, however this theorem can still be used in non-linear networks if only small changes in voltages and currents are considered: see the note about linearity and superposition for more details of the small-signal approximation.

current would flow in the external network X, no matter what X is (network X doesn't have to be linear). See the figure below:

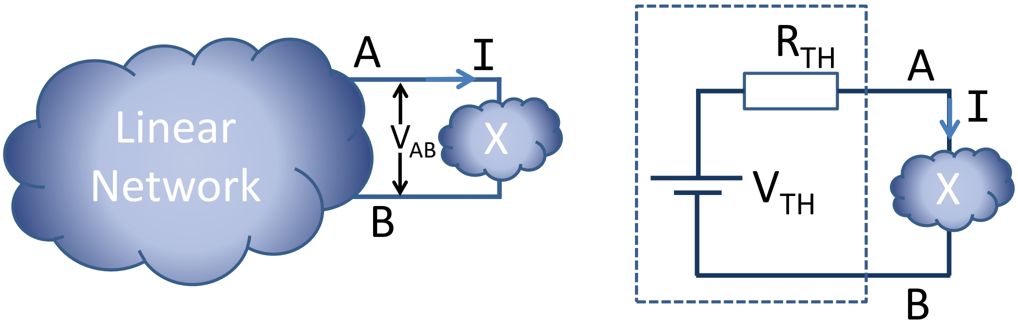

Figure 6.3 Thévenin's theorem: what we're trying to prove

For any particular external network X, there will be a current flowing out of the linear network, into X (shown as 'I' in the figure above), and a voltage across this network (shown as V_{AB}). The static resistance of the network X at this operating point is therefore V_{AB} / I.

Therefore, we can replace the external network with a current source of value I, and the voltage across the terminals AB should be the same: it will be whatever voltage is required to produce this current through the external network.

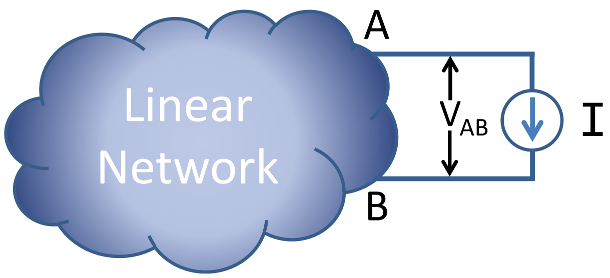

Figure 6.4 Proving Thévenin: replacing the external network

We now have a completely linear circuit, so we can apply superposition and consider that the voltage V_{AB} will be the sum of the voltages due to the external current source *I* and the voltage due to all of the internal voltage and current sources inside the linear network. To evaluate the first of these, set all of the voltage and current sources inside the linear network to zero. Since it is a linear network, the voltage V_{AB} must be linearly proportional to the current I, so no matter what the value of current is, we can write:

$$V_{AB\_1} = -kI \tag{6.3}$$

where *k* is a constant (I've included the negative sign in this equation since a positive current will cause the voltage at B to be higher than the voltage at A, and therefore the voltage V_{AB} to be negative; so including this factor of minus one means that *k* will be positive). Then the second contribution can be determined by setting the external current source to zero, and turning on all of the voltage and current sources in the linear network. This would then result in voltage across the terminals V_{AB_2}. The total voltage across the output is therefore:

$$V_{AB} = V_{AB\_1} + V_{AB\_2} = V_{AB\_2} - kI \tag{6.4}$$

and this will be true for any external source which passes a current *I* when a voltage V_{AB} is placed across it.

Now consider this network:

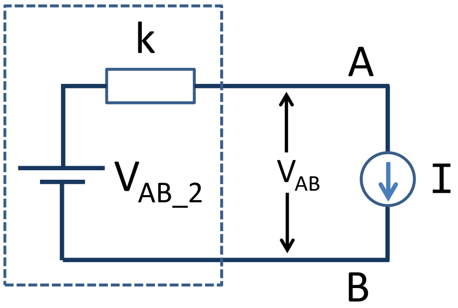

Figure 6.5 Proving Thevenin: the equivalent network

The voltage across the terminals AB in this network is given by:

$$V_{AB} = V_{AB\_2} - kI \tag{6.5}$$

But this is identical to equation (6.4). In other words, the equivalent network shown in Figure 6.5 produces the same voltage as the unknown linear network for any value of external current *I*.

Therefore the two networks are equivalent.

6.3.2 Determining the Thévenin equivalent network

Knowing that an equivalent network exists just leaves the problem of trying to work out what it is for any given network. How to do this depends on whether you are supplied with the circuit diagram of the linear network for analysis, or just presented with a box with two-terminals in the lab.

First, what do you do if presented with a circuit diagram?[4] The simplest approach is to consider what voltage would be measured if there was nothing connected to the two terminals, and then what current would flow if the two terminals were connected together:

[4] For what to do when measuring the equivalent network in the lab, see the last section in this chapter.

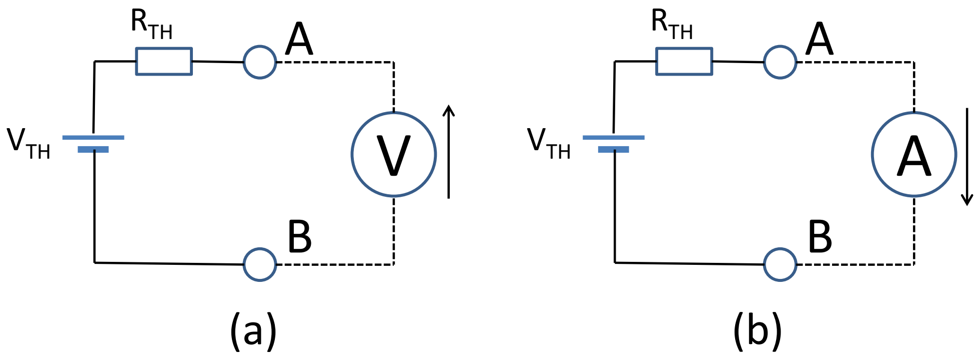

Figure 6.6 - Measuring the open-circuit voltage (a) and the short-circuit current (b)

With nothing[5] connected between the two terminals (see Figure 6.6(a)), no current will be flowing around the circuit, which means there is no voltage drop across the resistor[6], and if there is no drop across the resistor, the voltage measured on the output (known as the *open-circuit voltage*) must be equal to the voltage of the voltage source in the Thévenin equivalent network (otherwise known as the *Thévenin voltage*).

Next, you could short-circuit the two terminals[7] (see Figure 6.6(b)), and determine the current that flows between them (this is known as the *short-circuit current*). Effectively we're making the circuit shown in Figure 6.7:

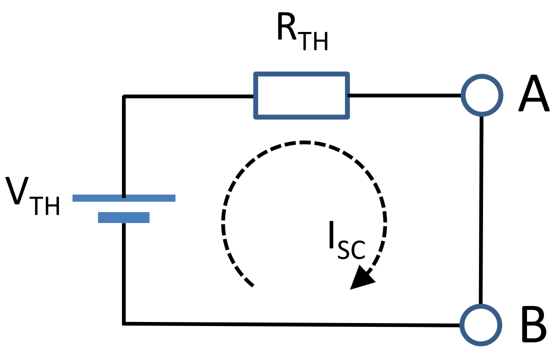

Figure 6.7 Producing the short-circuit current I_{SC}

and by Ohm's law, the current should be the voltage divided by the resistance, which in this case is the Thévenin voltage (which we already know since we've just worked it out) divided by the unknown resistance.

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[5] Note that an ideal voltmeter has an infinite resistance, so no current will flow through it. For now, we'll assume that the voltmeter used to measure the open-circuit voltage here is an ideal voltmeter.

[6] Applying Ohm's law to the resistors, V = IR, and if I (the current through the resistor) is zero, then V (the voltage across the resistor) must be zero as well.

[7] Note that an ideal ammeter has zero resistance, so there is no voltage drop across it, so the two output terminals are at the same voltage. For now, we'll assume that the ammeter used to measure the short-circuit current here is an ideal ammeter.

So we can calculate the resistance in the equivalent network just applying Ohm's law again:

$$R_{TH} = \frac{V_{TH}}{I_{SC}} \tag{6.6}$$

where R_{TH} is the resistance in the Thévenin equivalent network, V_{TH} is the voltage source in the Thévenin equivalent network, and I_{SC} is the current that flows between the terminals when they are connected together (the short-circuit current).

6.4 Norton's theorem

Norton's theorem states that for any two-terminal network containing only linear components and fixed voltage and current sources, there always exists an equivalent network that contains just one fixed current source, and one resistor wired in parallel.

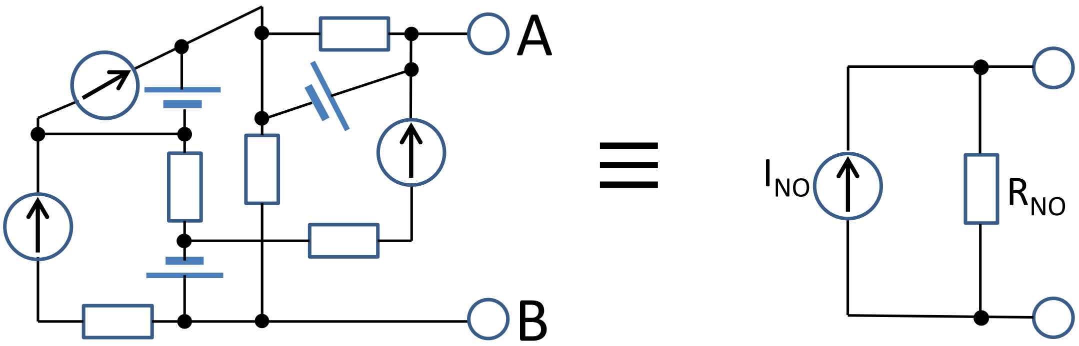

Figure 6.8 - Real and Norton equivalent network. (With the right choice of I_{NO} and R_{NO}, both of these networks would behave in the same way when the terminals A and B are connected to any other network.)

You can determine the Norton-equivalent current I_{NO} and the Norton-equivalent resistance R_{NO} from measuring the open-circuit voltage and the short-circuit current as well. The argument goes like this:

If nothing is connected to the terminals A and B, then all the current from the current source I_{NO} must flow through the resistor R_{NO}, so the voltage across the resistor, which will be equal to the open-circuit voltage, is:

$$V_{OC} = I_{NO} \times R_{NO} \tag{6.7}$$

If we directly connect A to B, then the voltage at point A will be equal to the voltage at point B. Therefore there will be no voltage across the resistor R_{NO} and hence no current through that resistor. So all of the current I_{NO} must flow between the terminals A and B, and that means that the short-circuit current measured would be:

$$I_{SC} = I_{NO} \tag{6.8}$$

From these two equations, it's quite easy to derive that:

$$
\begin{aligned}
I_{NO} &= I_{SC} \\
R_{NO} &= \frac{V_{OC}}{I_{NO}} = \frac{V_{OC}}{I_{SC}}
\end{aligned}
\tag{6.9}
$$

Notice that the Norton equivalent resistance and the Thévenin equivalent resistance are the same: they are both the ratio of the open-circuit voltage to the short-circuit current.

Proving Norton's theorem is simple once you've proved Thévenin's theorem: the Norton equivalent network is linear, and therefore it must have a Thévenin equivalent network. So anything which has a Thévenin equivalent network will also have a Norton equivalent network: the Norton network which has the same Thévenin equivalent network.

6.5 An example of working out the Thévenin equivalent network

This might all be clearer with an example. Consider the network shown below:

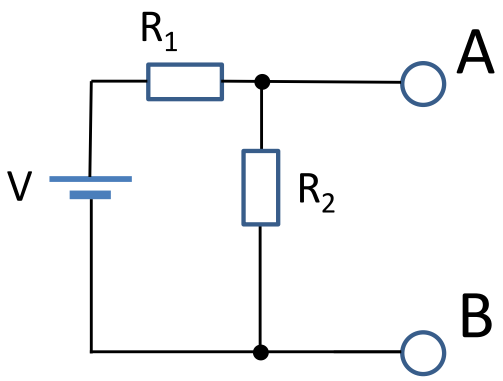

Figure 6.9 The Thévenin equivalent network for this is?

What is the Thévenin equivalent network for this network?

First step: determine the open-circuit voltage (the voltage with nothing connected to the terminals A and B). With nothing connected across the terminals this is just a potential divider, and the output voltage (the voltage across the R_2 resistor) is given by the standard potential divider equation:

$$V_{OC} = V\frac{R_2}{R_1 + R_2} \tag{6.10}$$

so that's the Thévenin voltage.

The next stage is to work out the short-circuit current (the current when the two terminals A and B are shorted together). If you do this with this circuit, the current would be given by:

$$I = \frac{V}{R_1} \tag{6.11}$$

since with A and B at the same voltage, all the voltage V must drop across the resistor R_1, and that results in a current V / R_1 flowing through R_1 and then out from terminal A and back in through terminal B.

The Thévenin resistance is then the ratio of the open-circuit voltage to the short-circuit current[8]:

$$R_{TH} = \frac{V_{OC}}{V/R_1} = \frac{V_{OC}}{V/R_1} = \frac{V\dfrac{R_2}{R_1+R_2}}{V/R_1} = \frac{R_1R_2}{R_1+R_2} \qquad (6.12)$$

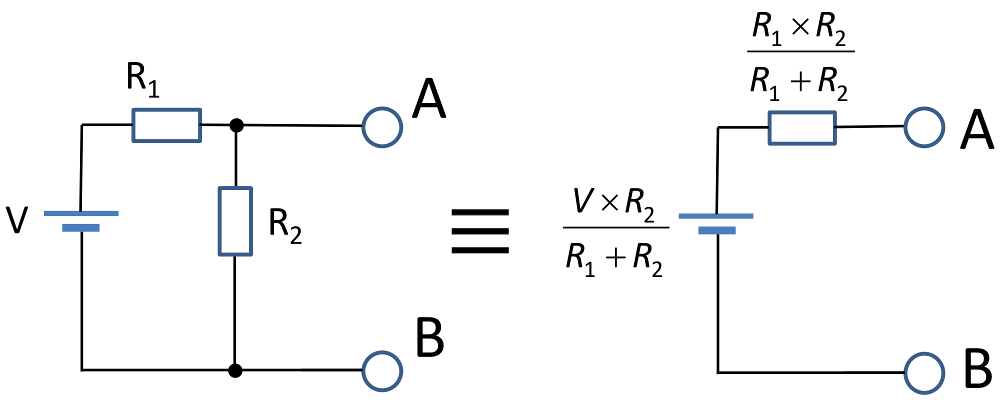

Figure 6.10 - A network with its Thévenin equivalent

It would be a good idea to get a lot of practice of this sort of analysis; it's a common exam question.

6.6 Determining the equivalent network resistance: a short cut

While the approach taken above to determine the equivalent network resistance for a Thévenin or Norton equivalent network (determining the open-circuit voltage and the short-circuit current and then dividing them) works fine, there's often a short cut you can use.

The theory goes like this: Thévenin and Norton's theorems only work for linear networks, and a linear network is one in which if you scale all the inputs by a factor *k*, all the outputs scale by the same factor *k*.

So: consider a network with a Thévenin equivalent network, like that shown in Figure 6.10. If the voltage V was doubled, then the voltages everywhere in the network would be doubled, and the currents everywhere would be doubled. That means the open-circuit voltage would be doubled, and the short-circuit current would be doubled, and that means that the equivalent network resistance doesn't change at all:

$$R_{TH} = \frac{2 \times V_{OC}}{2 \times I_{SC}} = \frac{V_{OC}}{I_{SC}} \qquad (6.13)$$

The same is true if the voltages and currents in the network are multiplied by any constant term. So, consider if they are multiplied by the constant zero: all the voltage sources would become zero volts, and all the current sources would become zero amps, but the equivalent network resistance still doesn't change[9].

[8] You might note that this is equal to the parallel combination of R_1 and R_2. This is not a co-incidence.

[9] At this point a mathematician might point out that if the short-circuit current was zero, then the equivalent resistance is not defined, since you can't divide by zero. To which an engineer might reply that instead of

Putting a zero volt source between two points in a network means that you are forcing the voltage at those two points to be equal. In other words, it's just like putting a wire link between them.

Putting a zero amp current source between two points in a network means that no current at all is flowing through the current source. In other words, it might as well not be there, it's not doing anything useful.

So, to work out the resistance in the equivalent network, you can replace all of the voltage sources with wires and remove all the current sources entirely, and then work out what the resistance would be between the two terminals. In the case of the network above, this would produce:

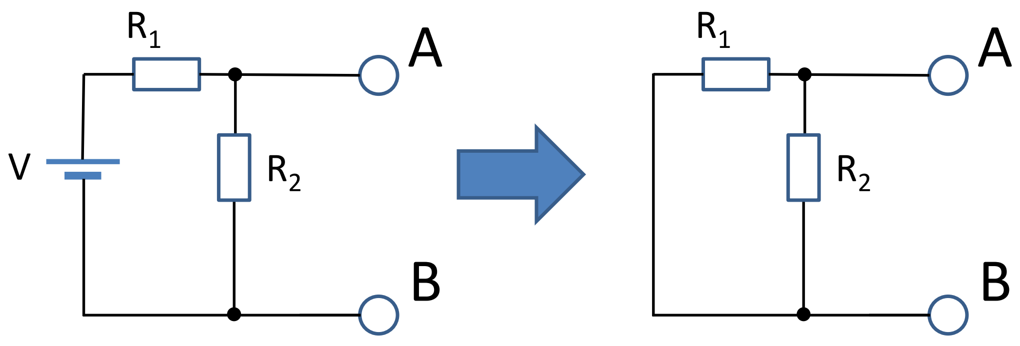

Figure 6.11 Setting the voltage source to zero

From this you can see by inspection that the resistance between A and B (which is equal to the equivalent network resistance) is just the two resistors R_1 and R_2 in parallel.

The technique is even more powerful in more complex networks, for example consider the network on the left in Figure 6.12 below:

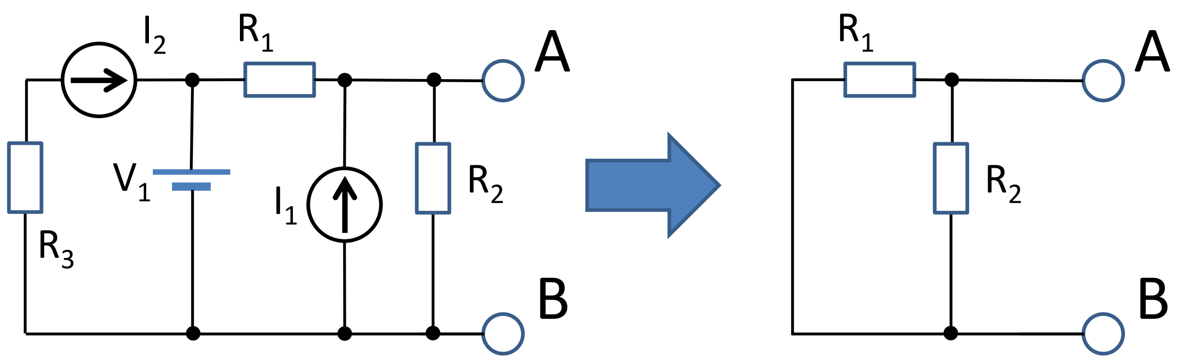

Figure 6.12 A more complex network, setting both voltage and current sources to zero

It looks more complex than the network in Figure 6.11, and it would certainly be more difficult to work out the open-circuit voltage or the short-circuit current, but if all you wanted to know was the equivalent network's resistance, it's just as easy. Set all the voltage sources to a short-circuit (a

multiplying each source by zero, they could be multiplied by $10^{-1,000,000}$, which to all practical purposes is the same thing.

wire) and all the current sources to an open-circuit (remove them entirely) and the network collapses to exactly the same network as before[10].

6.6.1 One important consideration

I wrote that this was a technique that could "often" be used. It doesn't always work, and it's important to know when you can't use it. It often doesn't work when there are dependent sources in the network.

For example, consider the following network:

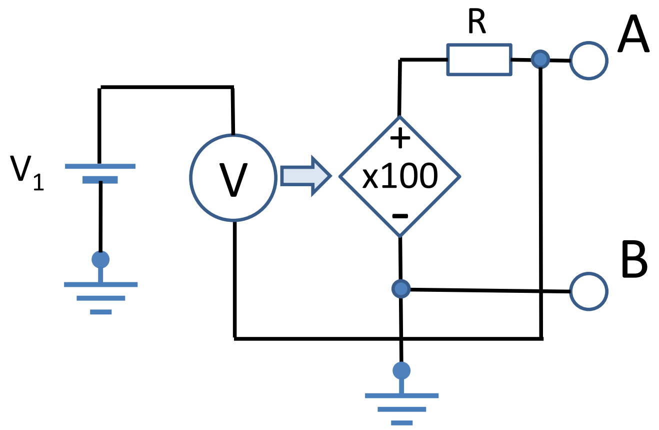

Figure 6.13 A dependent source with feedback

This network features a dependent source, which creates a voltage one hundred times greater than the voltage reading on the voltmeter.

Analysis of this network shows that the open-circuit voltage is given by:

$$
\begin{aligned}
100\left(V_1 - V_{OC}\right) &= V_{OC} \\
V_{OC} &= \frac{100}{101} V_1
\end{aligned}
\tag{6.14}
$$

and the short-circuit current (when the output is at zero volts) would be:

$$I_{SC} = \frac{100 V_{in}}{R} \tag{6.15}$$

and this suggests a Thévenin output resistance of:

$$R_{TH} = \frac{V_{OC}}{I_{SC}} = \frac{100V_1}{101} \frac{R}{100V_1} = \frac{R}{101} \tag{6.16}$$

[10] There's no need to consider R3, since with the current source on the left set to zero, there can be no current flowing through this resistor, so it might as well not be there.

However, if you just set the independent voltage source to zero, then the dependent voltage source would also be at zero volts, and looking into the terminal A, an external network would see a resistor of value R apparently connected to ground. However this is not the Thévenin resistance in this case.

The difference is due to the dependent source: as soon an external network attempts to send any current into the network through the resistance R, the voltage across the voltmeter, and hence the dependent voltage source, would change. The dependent voltage source would no longer be at zero volts.

6.7 Determining the open-circuit voltage: a short cut

If you are faced with the network shown in Figure 6.12 above with and asked to derive its equivalent, then while the equivalent network resistor is easy to determine, what about the open-circuit voltage?

The easiest approach here is to take advantage of the fact that the network is linear, and use superposition[11]. Superposition allows us to consider each source separately, and then combine the results. Here, that would mean first setting I_1 and I_2 to zero, which reduces the network to:

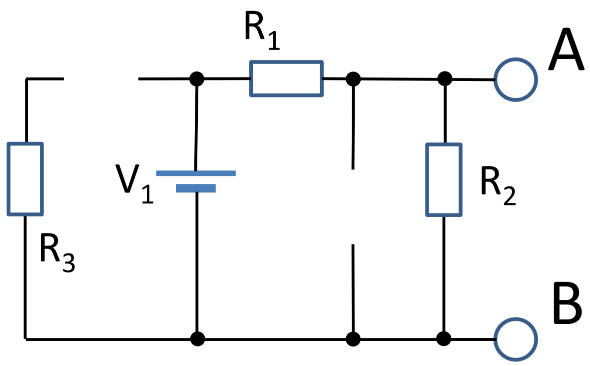

Figure 6.14 More complex network with only the voltage source considered

Setting the current sources to zero (which effectively means removing them from the network entirely) reveals a potential divider, and an output voltage of:

$$V_{out\_V1} = V_1 \frac{R_2}{R_1 + R_2} \tag{6.17}$$

Next, consider just the first current source I1, setting the voltage source and second current source to zero (noting again that a zero voltage source is equivalent to a wire, and a zero current source is equivalent to an open circuit). This reduces the network to:

[11] See the chapter about linearity and superposition for more details on this point.

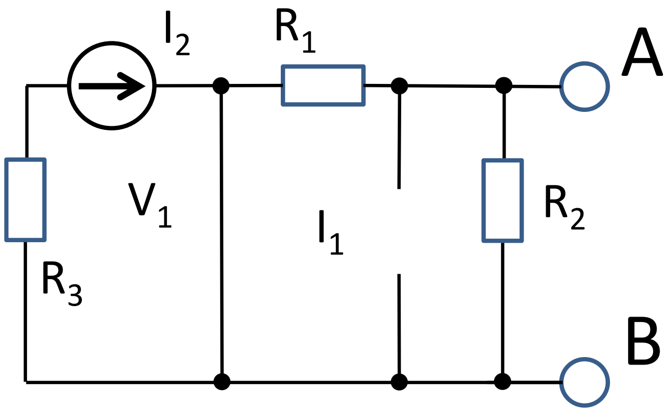

Figure 6.15 More complex network with only one current source considered

and this suggests an output voltage of zero, since all the current would flow through the wire that used to be the voltage source, rather than the two resistors R_1 and R_2. With no current through R_2, the voltage across it (which is the output voltage) will be zero.

Finally, consider the second current source only:

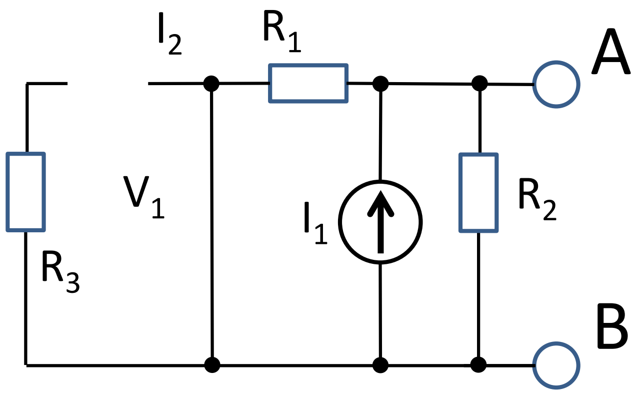

Figure 6.16 More complex network with the other current source set to zero

Here the current from I_1 will split, with a proportion $R_1 / (R_1 + R_2)$ going through R_2, and a proportion $R_2 / (R_1 + R_2)$ going through R_1[12]. That means the output voltage is:

$$V_{out\_I1} = I_1 \frac{R_1}{R_1 + R_2} \tag{6.18}$$

Using the principle of superposition, the total output voltage for the whole network is:

$$V_{out} = V_{out\_V1} + V_{out\_I1} + V_{out\_I2} = V_1 \frac{R_2}{R_1 + R_2} + 0 + I_1 \frac{R_1}{R_1 + R_2} = \frac{V_1 R_2 + I_1 R_1}{R_1 + R_2} \tag{6.19}$$

and this is often easier than trying to solve the whole network at once.

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[12] See the note on resistor networks for more information on where this result comes from.

6.8 Determining the equivalent network in the lab

As well as speeding up the analysis of circuits, Thévenin's theorem can be very useful in the lab. Determining the equivalent network for something like an amplifier allows you to determine what will happen when that amplifier is connected to any particular load.

However, there's a catch: while it's usually possible to measure the open-circuit voltage between two points with reasonable accuracy, measuring the short-circuit current can introduce some practical problems: it usually isn't a good idea to short-circuit the outputs of a network. Amplifiers in particular tend to get very hot if you do that, and they are only specified to work with a certain minimum load resistance. (There is often a minimum load specified for amplifiers, otherwise they leave their linear region of operation, and once they start behaving in a non-linear way, you can't define a Thévenin equivalent network at all.)

Fortunately, there's another way to do it. Consider the following circuit:

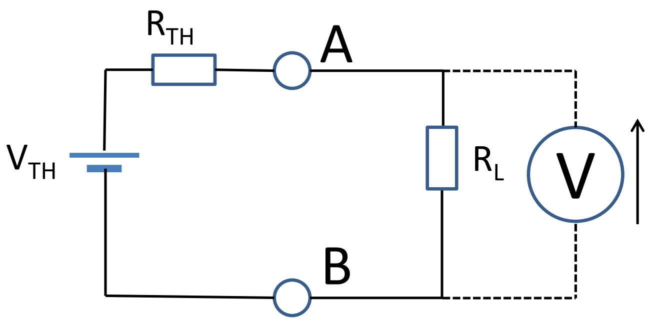

Figure 6.17 - Measuring the Thévenin resistance using a load resistor

We can measure the open-circuit voltage (by setting R_L to infinity, see above), and we also know the resistance of the load resistor on the output (R_L in Figure 6.17), since we just put it there. This circuit is now just a potential divider, so the output voltage will be given by the standard potential divider equation:

$$V_{out} = V_{TH} \frac{R_L}{R_{TH} + R_L} \tag{6.20}$$

where again I've used the symbols V_{TH} as the Thévenin voltage, R_{TH} as the Thévenin resistance, and R_L as the load resistance that we've connected to the output terminals. Measure the output voltage, and a bit of algebra reveals that:

$$R_{TH} = \frac{(V_{TH} - V_{out})R_L}{V_{out}} \tag{6.21}$$

If you're using a variable resistor as the load resistor R_L, there's an even easier way: consider what happens if you adjust the load resistor until the output voltage V_{out} is exactly one-half of the open-circuit voltage V_{TH}:

$$R_{TH} = \frac{(V_{TH} - V_{TH}/2)R_L}{V_{TH}/2} = \frac{(V_{TH}/2)R_L}{V_{TH}/2} = R_L \tag{6.22}$$

If you can adjust the load resistance until the output voltage is one-half of the open-circuit output voltage (the output voltage with no load connected), then at that point, the load resistance will be equal to the Thévenin resistance.

Unfortunately some amplifiers[13] have a very low output resistance (Thévenin resistance), and don't like driving equally low resistance loads since this would exceed the maximum current they can supply, so you have to use the technique described in equation (6.21) with the load resistance rather higher than the output resistance.

6.8.1 The recommended method for determining equivalent networks in the lab

While simple, the technique described above is not the recommended approach if you have a bit more time and want to get a more accurate estimation of the equivalent network. There's a better way, which also avoids the practical problem that it's impossible to determine the open-circuit voltage directly, since all real voltmeters have a finite resistance.

Consider the equivalent network of any provided unknown system, connected to a variable load resistor. I'll assume here the most common case in the lab, which is when one terminal of the two-terminal network is grounded (connected to zero volts), and only consider the other one (called the *output* here).

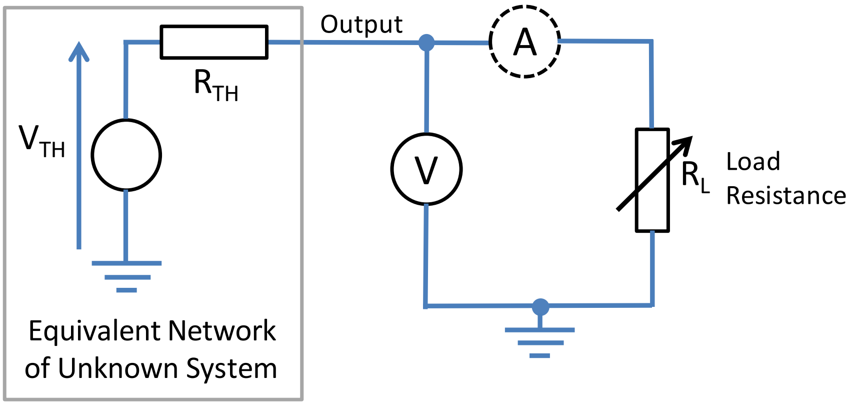

Figure 6.18 Determining the equivalent network in the lab

The voltmeter is measuring the mid-point of a potential divider, with the equivalent voltage source V_{TH} (the open-circuit output voltage) driving a series combination of the equivalent output resistance R_{TH} (the output resistance) and the load resistance R_L. Re-drawn (and without the ammeter, since you don't really need it: you can work out the current knowing the voltage across the voltmeter and the resistance R_L), it looks more like the familiar potential divider:

[13] For example the output of an op-amp amplifier.

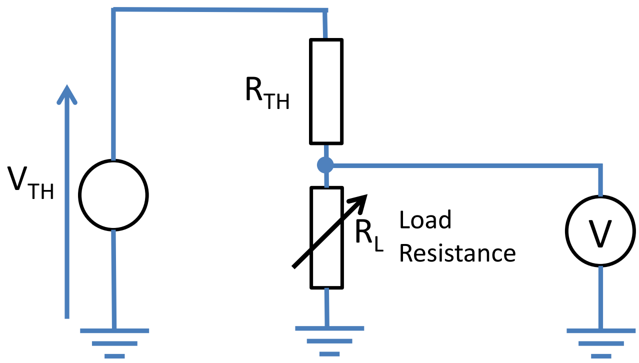

Figure 6.19 A potential divider version of the lab-measurement circuit

The voltage measured by the voltmeter is therefore given by:

$$V = V_{TH} \frac{R_L}{R_{TH} + R_L} \tag{6.23}$$

and the current through the load resistance is:

$$I = \frac{V}{R_L} = \frac{V_{TH}}{R_{TH} + R_L} \tag{6.24}$$

Eliminating R_L from these equations gives:

$$\begin{aligned} V &= V_{TH} \frac{V/I}{R_{TH} + V/I} \\ 1 &= V_{TH} \frac{1}{R_{TH} \times I + V} \\ R_{TH} \times I + V &= V_{TH} \\ V &= V_{TH} - R_{TH} \times I \end{aligned} \tag{6.25}$$

which suggests that plotting the measured values of *V* (on the vertical axis) against the measured values of *I* (on the horizontal axis) should give a straight line[14], with a gradient of $-R_{TH}$.

[14] Occasionally, I get asked why, if we're plotting voltage against current, isn't the gradient of the curve the resistance? The answer: because we're not plotting the voltage across the resistor against the current through the same resistor. If we were doing that then yes, we would expect to see a straight line through the origin which had the resistance as the gradient. But we're not: we're plotting the voltage across the source V_{TH} minus the voltage across the resistor R_{TH} against the current through R_{TH}, and that subtraction leads to a graph where the resistance R_{TH} is the negative gradient of the graph.

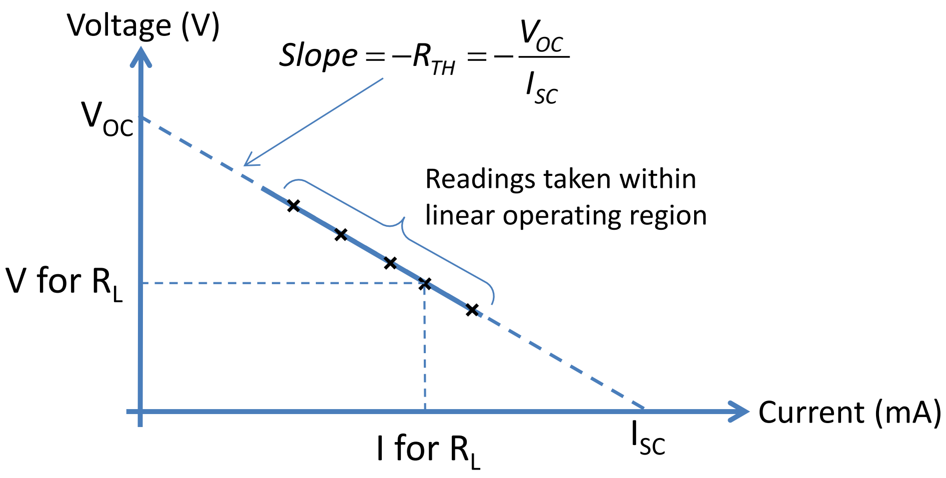

Figure 6.20 Plotting measurements to determine the equivalent network in the lab

This is a better way to do things in practice, since it has several key advantages:

- It does not require measurement of the open-circuit voltage, so it can be used with a real voltmeter with a finite resistance (you just calculate the effective load resistor to be the added resistor R_L in parallel with the resistance of the voltmeter).
- It does not require direct measurement of the short-circuit current, which in many cases is not practical due to networks being forced outside their linear range of operation (and potentially damaged as a result).
- It is clear whether the network is behaving as a linear network or not: if the points plotted do not lie on a straight line, the network is not behaving as a linear network and no equivalent network can be determined.
- It does not rely on one to two single measurements (which might have errors in them). If several points are taken, and they all lie on a straight line, there can be more confidence that the network is behaving as expected.

6.9 Summary: the most important things to know

- Linear two-terminal networks can be replaced with simple equivalent network consisting of a voltage source and series resistor (Thévenin) or current source and parallel resistor (Norton).
- The Thévenin voltage is the open-circuit voltage; the Norton current is the short-circuit current. In both cases the resistor is the ratio of the open-circuit voltage to the short-circuit current.