5 A Short Introduction to Linearity and Superposition

v1.5 – June 2021

Prerequisite knowledge required: Diodes, Ohm and Kirchhoff's Laws

5.1 Introduction and definitions

Many electronic circuits have more than one input, and in these cases the output can be a complex function of all of the inputs, and difficult to work out. However, in there is a short-cut that can be used in many situations which can make the output much easier to calculate (or at least approximate). This is the principle of superposition and it's the subject of this chapter.

The circuits in which this short-cut can be taken are known as "linear circuits". It's important to be clear about what is meant by "linear" in this context: a DC circuit is linear if the output y is related to the inputs x_1, x_2, x_3 by a formula of the type:

$$y = c_1 x_1 + c_2 x_2 + c_3 x_3 + \ldots = \sum_i c_i x_i \tag{5.1}$$

where c_1, c_2, etc are constants (note that these inputs and output can be either currents or voltages or a mixture of both – for example Ohm's law is linear since with an input current and output voltage, Ohm's law gives output = input * resistance).

A couple of consequences of this definition are worth noting:

- If all inputs are increased by a factor of k, then the output will also be increased by the same factor k
- The output when all inputs are present is equal to the sum of the outputs when each input is present with all other inputs set to zero.

This last consequence leads to the principle of superposition: "for any linear circuit with multiple inputs, the output resulting from a number of independent inputs is equal to the sum of the outputs produced by each independent input taken in turn."

For example, consider a circuit with two inputs, A and B. If the output is a linear function of A and B, then the output when both A and B are present is equal to the output when A is present and B is zero plus the output when B is present and A is zero.

5.1.1 A mathematical aside

For superposition to work, it can be shown that there is only one operation that a DC circuit can do on an input to create the output: multiply the input by a constant[1]. Anything else that a circuit does (for example adding a constant or squaring the input voltage) means that the circuit is not linear, and hence that the sum of the output with two different inputs will not be equal to the output when the input is the sum of the inputs.

[1] For AC circuits (in which the voltages are functions of time) there are a few more possible operations that retain the property of superposition: differentiation, integration and introducing a time delay.

For example: consider a circuit with two inputs which produces an output by adding one volt to the sum of the inputs to form the output. This is not a linear circuit, as the output cannot be expressed in terms of the inputs using equation (5.1). The behaviour of the equation can be expressed mathematically as:

$$V_{out} = V_1 + V_2 + 1 \tag{5.2}$$

The result from having an input of V_1 present when $V_2 = 0$ is:

$$V_{out\_1} = V_1 + 1 \tag{5.3}$$

and the result when $V_1 = 0$ but V_2 is present is:

$$V_{out\_2} = V_2 + 1 \tag{5.4}$$

In this case adding the outputs from the input voltages V_A and V_B would be:

$$V_{out} = (V_1 + 1) + (V_2 + 1) = V_1 + V_2 + 2 \tag{5.5}$$

However, the output from having both inputs present at the same time is:

$$V_{out} = (V_1 + V_2) + 1 = V_1 + V_2 + 1 \tag{5.6}$$

which is clearly not the same thing. Superposition is not working for this non-linear circuit.

However, consider a circuit that instead of adding one volt, multiplies the sum of the inputs by a factor of two. Then the result from only having the input V_1 is:

$$V_{out\_1} = 2 \times V_1 \tag{5.7}$$

and the result from only having the input V_2 is:

$$V_{out\_2} = 2 \times V_2 \tag{5.8}$$

so that in this case the sum of the outputs for each input in turn is:

$$V_{out} = (2 \times V_1) + (2 \times V_2) = 2 \times V_1 + 2 \times V_2 \tag{5.9}$$

For this circuit, the output from having both inputs present at the same time is:

$$V_{out} = 2 \times (V_1 + V_2) = 2 \times V_1 + 2 \times V_2 \tag{5.10}$$

and this time these two are equal, so this is a linear circuit (and the output can be expressed in terms of equation (5.1)).

5.2 Superposition and linearity in circuits

At the risk of over-emphasising this point: the principle of superposition only applies to linear circuits (and a linear circuit is one in which increasing all the inputs by a factor of *X* also increases the output by a factor of *X*).

For example, consider a circuit with a voltage source, a resistor and an ammeter measuring the current, something like this:

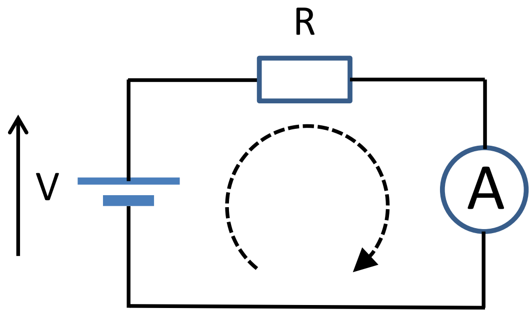

Figure 5.1 Simple circuit with voltage input and current output

Define the voltage source as the input, and the current measured by the ammeter as the output. Then as long as the ammeter is ideal (so it has zero resistance) we can apply Ohm's law, and write:

$$I = \frac{V}{R} \tag{5.11}$$

This is a linear system, since it can be expressed in terms of equation (5.1) with the constant c_1 equal to 1/R. Doubling the input will produce an output current I_2 which is double the original current I:

$$I_2 = \frac{2V}{R} = 2\frac{V}{R} = 2 \times I \tag{5.12}$$

However, suppose that the element in the circuit is not a resistor, but a diode. Then the current flowing would be given by[2]:

$$I = I_0\left(\exp\left(\frac{eV}{nkT}\right) - 1\right) \tag{5.13}$$

which is not in the form of equation (5.1), and now doubling the input (the voltage) results in a current of:

$$I_2 = I_0\left(\exp\left(\frac{2eV}{nkT}\right) - 1\right) \tag{5.14}$$

[2] This is the Shockley diode equation, for more details see the introduction to diodes.

and this is most definitely not equal to $2 \times I$. The circuit is no longer linear: diodes are non-linear devices, so the principle of superposition does not apply here. This makes circuits including diodes much harder to analyse.

5.3 Circuit analysis using superposition

Just how useful this principle is can probably best be shown by doing an example with and without it. Consider the circuit shown below, where you want to know the current through the resistor R_1:

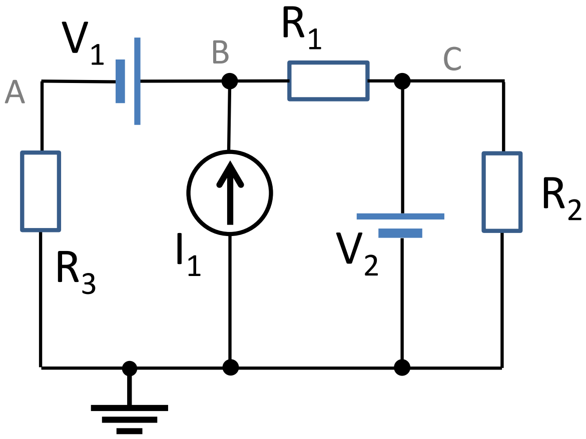

Figure 5.2 A linear circuit with three sources and four nodes

Doing this without superposition would require solving the circuit, which requires several steps. For example, labelling the nodes not at ground in the circuit A, B and C (as shown in the figure above) allows us to write three equations in these three unknowns.

First, note that V_1 and V_2 define the potential between two of the nodes, so:

$$B - A = V_1 \tag{5.15}$$

$$C = V_2 \tag{5.16}$$

and applying Kirchhoff's current law to node B (and Ohm's law to R_1 and R_3) reveals:

$$\frac{A}{R_3} + \frac{B - C}{R_1} = I_1 \tag{5.17}$$

Then these three equations can be solved by substitution. First eliminate A between equations (5.17) and (5.15):

$$\frac{B - V_1}{R_3} + \frac{B - C}{R_1} = I_1 \tag{5.18}$$

and then eliminate C between equations (5.16) and (5.18):

$$\frac{B - V_1}{R_3} + \frac{B - V_2}{R_1} = I_1 \tag{5.19}$$

which after a bit of tedious algebra reveals that:

$$B = \frac{I_1 + \frac{V_1}{R_3} + \frac{V_2}{R_1}}{\frac{1}{R_1} + \frac{1}{R_3}} \tag{5.20}$$

so the current through the resistor must be:

$$I_{R1} = \frac{\frac{I_1 + \frac{V_1}{R_3} + \frac{V_2}{R_1}}{\frac{1}{R_1} + \frac{1}{R_3}} - V_2}{R_1} \tag{5.21}$$

5.3.1 Or by superposition…

Doing the same calculation by superposition is somewhat easier. The idea is that we can calculate any linear parameter of the circuit (in this case the current through the resistor R_1) one independent source at a time, and then just add up the answers.

In this case, first consider V_1, and set the sources I_1 and V_2 to zero. This effectively converts the circuit into:

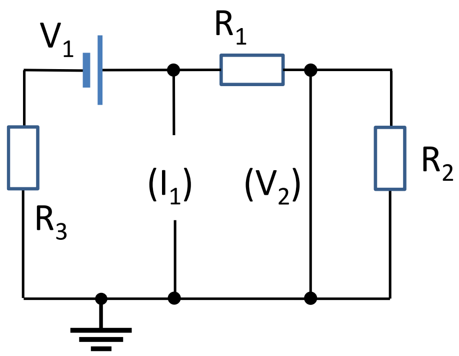

Figure 5.3 A linear circuit with only one source considered

In this case the current through R_1 is easy to determine: the equivalent resistance from one side to V_1 to the other is just $R_1 + R_3$, so the current through R_1 must be $I_{V1} = V_1 / (R_1 + R_3)$.

Next, consider the current source only, setting the two voltage sources to zero. The circuit has effectively become:

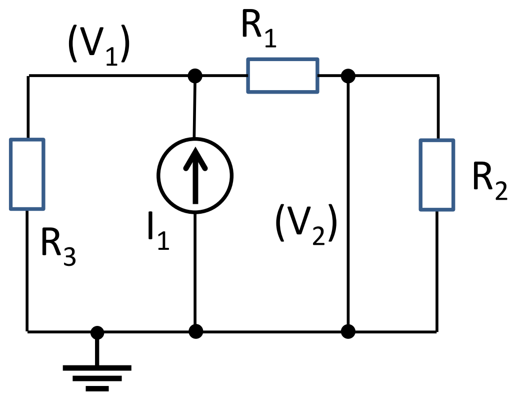

Figure 5.4 A linear circuit with only the current source considered

The current from I_1 will be split between R_1 and R_3, and the standard formula for current through two parallel resistors gives that the current flowing through R_1 must be:

$$I_{I1} = I_1 \frac{R_3}{R_1 + R_3} \tag{5.22}$$

Finally considering the second voltage source only, setting the other two sources to zero, gives the effective circuit:

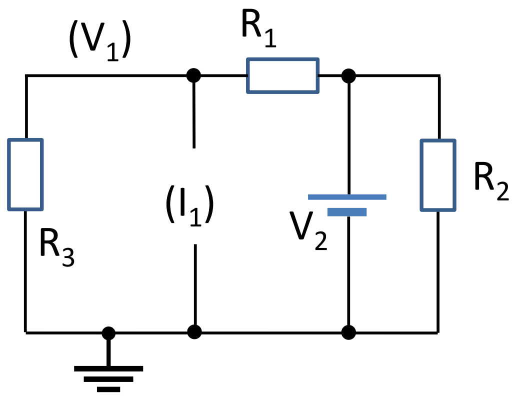

Figure 5.5 A linear circuit with only the second voltage source considered

Here the current flowing through R_1 is clearly:

$$I_{V2} = \frac{V_2}{R_1 + R_3} \tag{5.23}$$

The principle of solution tells us that the total current through the resistor R_1 must be the sum of these three currents. All we need to do is be careful that we add up all the currents going in the same direction, which is an issue here, since V_1 and I_1 send current through R_1 from left to right in the diagrams above, whereas V_2 sends current through from right to left. So the total current flowing from left to right is:

$$
\begin{aligned}
I_{R1} &= I_{V1} + I_{I1} - I_{V2} = \frac{V_1}{R_1 + R_3} + I_1 \frac{R_3}{R_1 + R_3} - \frac{V_2}{R_1 + R_3} \\
&= \frac{V_1 + I_1 R_3 - V_2}{R_1 + R_3}
\end{aligned}
\tag{5.24}
$$

Not only is the calculation easier, but it's led to a neater way of expressing the result[3].

5.4 Linearity and small signals

Linear techniques (and hence superposition) are so useful and easy to use that they are commonly used to produce approximations to the performance of non-linear circuits as well. This technique is called the *small-signal approximation*.

The idea is that even if a circuit is non-linear, provided the input doesn't change very much, then for the small changes in the output caused by small changes in the inputs the circuit behaves approximately like a linear circuit. (This is often useful in practice, as all real circuits are non-linear to some extent, and yet circuits are designed to be used with only small signals.)

For example, consider the following circuit:

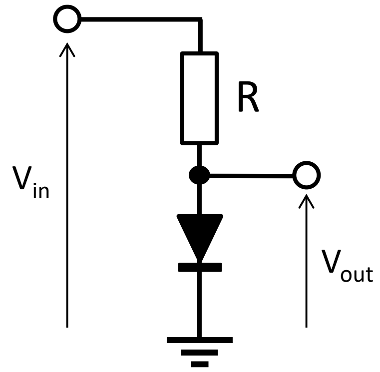

Figure 5.6 A non-linear potential divider circuit

The circuit is clearly non-linear: double the input and while the output will increase slightly it certainly won't double; for most diodes it's likely to stay between 500 mV and 800 mV for a very wide range of values of V_{in}.

If you plot the response of this circuit (V_{out} as a function of V_{in}) you get a graph something like this:

[3] I'll leave the task of showing that the expressions derived in the two cases (with and without superposition) are indeed equivalent as an exercise for the interested reader.

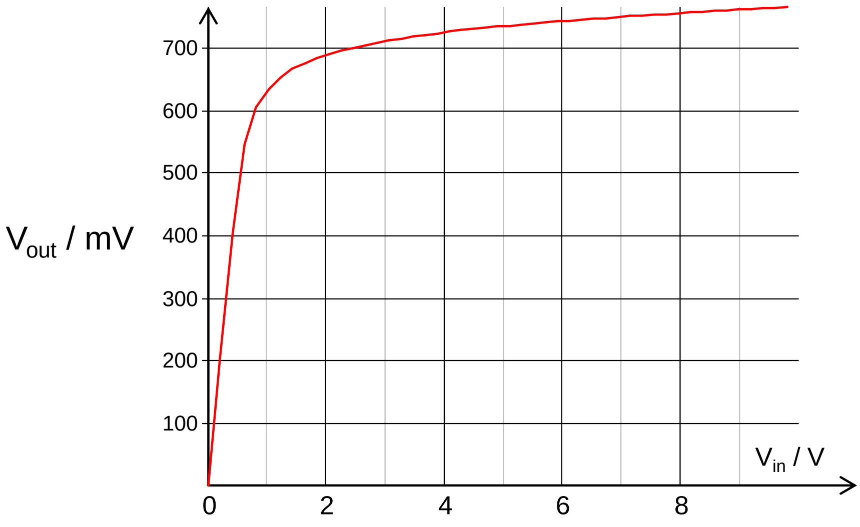

Figure 5.7 Non-linear response of a diode-resistor potential divider

which is clearly non-linear[4]. But what if you were only interested in signals that were around four volts? Looking at just this section of the graph, we'd notice that around this *operating point*, the graph can be closely approximated by a straight line:

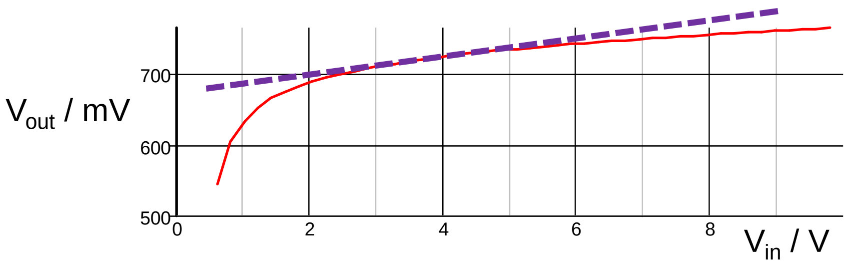

Figure 5.8 Linear approximation to non-linear response for the diode-resistor potential divider

Now supposing that the input signal V_{in} was expressed as:

$$V_{in} = 4 + \Delta V_{in} \tag{5.25}$$

where ΔV_{in} is a small change in the input voltage; in other words ΔV_{in} is the (small) difference between the actual input and four volts:

$$\Delta V_{in} = V_{in} - 4 \tag{5.26}$$

And suppose that the output was expressed as:

$$V_{out} = 0.72 + \Delta V_{out} \tag{5.27}$$

in other words ΔV_{out} is the (small) difference between the actual output and 0.72 volts.

$$\Delta V_{out} = V_{out} - 0.72 \tag{5.28}$$

[4] A linear response would be a straight-line going through the origin on a graph of input against output.

The straight line approximation to the non-linear response at this point can be written as:

$$V_{out}\left(\mathrm{mV}\right)=680+10\times V_{in} \tag{5.29}$$

which can also be written (using equation (5.25)) as:

$$\begin{aligned} V_{out} &= 0.68+0.01\times\left(4+\Delta V_{in}\right) \\ &= 0.72+0.04\times\Delta V_{in} \\ V_{out}-0.72 &= 0.04\times\Delta V_{in} \\ \Delta V_{out} &= 0.04\times\Delta V_{in} \end{aligned} \tag{5.30}$$

This now looks like a linear response: any small change in the input ΔV_{in} is multiplied by a constant factor of 0.04 to become a change in the output voltage ΔV_{out}. So this non-linear circuit does behave in a linear way provided it is the small changes in the input and output voltage that are considered, and not the entire input and output voltages.

We can show this working by considering an input that is the sum of two small voltages (plus the offset of 4 volts). So we can write:

$$V_{in}=4+\Delta V_{in}=4+\Delta V_1+\Delta V_2 \tag{5.31}$$

Because we are approximating the characteristic as a straight line, we could then write:

$$\begin{aligned} V_{out}\left(\mathrm{mV}\right) &= 720+\Delta V_{out}=720+10\times\Delta V_{in}=720+10\times\left(\Delta V_1+\Delta V_2\right) \\ \Delta V_{out} &= 10\times\left(\Delta V_1+\Delta V_2\right) \\ &= 10\times\Delta V_1+10\times\Delta V_2 \end{aligned} \tag{5.32}$$

and in this sense the system is called linear. Provided the voltage across the device doesn't move very far from one point (known as the operating point), the system can be approximated to be linear, and two small deviations from the operating point produce at the output the sum of the deviations caused by both inputs separately.

This sort of approach is very common in circuit analysis, and is called a *small-signal model*. I'll say a bit more about it.

5.5 Small-signal models

Just about any circuit is linear (once any constant DC offsets are removed) if the input signals are small enough; conversely just about any circuit is non-linear if the input signals are too large. It's very common when simulating non-linear circuits to first analyse the circuit at DC to work out what the average DC levels at each point in the circuit are (this is known as the *bias point* and this calculation has to be done using the full non-linear equations describing each component: this can be difficult). Having done this, a linear small-signal model can then be produced, and this is then used to work out the response of the system to the small signals it will be asked to deal with (which is much easier).

The results obtained are only approximate, but it's an approximation which is often accurate enough for small signals, especially when the circuit has been designed to be almost linear.

For example, consider the non-linear potential divider considered earlier (see Figure 5.6).

Since the current through the resistor and the current through the diode must be equal, we can use the Shockley diode equation with Ohm's law and Kirchhoff's current law to reveal the relationship between V_{out} and V_{in} for this circuit:

$$\frac{V_{in} - V_{out}}{R} = I_S \left(e^{\frac{eV_{out}}{kT}} - 1 \right) \tag{5.33}$$

This is not a particularly easy equation to work with, and finding the output voltage V_{out} for any given input voltage V_{in} has to be done using numerical iteration techniques. It's time-consuming, and not something you want to have to repeat every time the input voltage changes slightly.

However, once we know the value of V_{out} for one particular value of V_{in}, we can express the relationship between any small changes in V_{in} and V_{out} using linear small-signal methods. The first step is to find a suitable approximate linear model, and for that we differentiate equation (5.33) with respect to V_{in}. This reveals:

$$\begin{aligned}
\frac{1}{R} - \frac{1}{R}\frac{dV_{out}}{dV_{in}} &= I_S \frac{e}{kT} e^{\frac{eV_{out}}{kT}} \frac{dV_{out}}{dV_{in}} \\
\frac{dV_{out}}{dV_{in}} \left(I_S \left(\frac{e}{kT} \right) e^{\frac{eV_{out}}{kT}} + \frac{1}{R} \right) &= \frac{1}{R} \\
\frac{dV_{out}}{dV_{in}} &= \frac{\left(\frac{kT}{eI_S} \right) e^{\frac{-eV_{out}}{kT}}}{R + \left(\frac{kT}{eI_S} \right) e^{\frac{-eV_{out}}{kT}}}
\end{aligned} \tag{5.34}$$

This produces the gradient of the plot of the V_{out} / V_{in} characteristic of this circuit. I've written it this way so that it can easily be compared to the standard formula for a potential divider:

$$\frac{V_{out}}{V_{in}} = \frac{R_2}{R_1 + R_2} \tag{5.35}$$

which when differentiated, would give:

$$\frac{dV_{out}}{dV_{in}} = \frac{R_2}{R_1 + R_2} \tag{5.36}$$

Effectively, for small changes in V_{in} and V_{out} the resistor-diode potential divider is behaving like a standard potential divider, only with the diode having an effective resistance at this operating point of:

$$R_{eff} = \left(\frac{kT}{eI_S}\right) e^{\frac{-eV_{out}}{kT}} \qquad (5.37)$$

This is known as the *dynamic resistance* of the diode: it's the ratio of the small change in voltage across the diode to the small change in current through it, and it determines how the diode behaves around this operating point.

Now for small changes ΔV_{in} and ΔV_{out}, we can approximate:

$$\frac{\Delta V_{out}}{\Delta V_{in}} \approx \frac{dV_{out}}{dV_{in}} = \frac{1}{R \times I_S \left(\frac{e}{kT}\right) e^{\frac{eV_{out}}{kT}} + 1} \qquad (5.38)$$

and provided that the changes in the voltages are small, the right-hand-side of this equation can be considered to be a constant, with a value determined by the voltage and current at the operating point.

So a change in V_{in} of, say, 1 mV would be expected to produce a change in V_{out} of:

$$\Delta V_{out} = \frac{0.001}{R \times I_S \left(\frac{e}{kT}\right) e^{\frac{eV_{out}}{kT}} + 1} \qquad (5.39)$$

Putting in some typical values (I_S = 2 nA, R = 1k, T = 290 K and V_{out} = 0.6 Volts), reveals that:

$$\Delta V_{out} = \frac{0.001}{1000 \times 2 \times 10^{-9} \left(\frac{1}{0.025}\right) e^{\frac{0.6}{0.025}} + 1} = 472 \text{ pV} \qquad (5.40)$$

and since this is assumed to be a linear system (at least for small values of input voltage) we can immediately conclude that the change in output for a change in input of 2 mV would be 944 pV, and so on.

That's a lot easier than going back to the full non-linear equation (5.33) and trying to solve it again for a slightly increased value of V_{in}.

5.6 Summary: the most important things to know

- The principle of superposition allows circuits with multiple sources to be more easily analysed by considering the effect of each source in turn and summing these effects.
  - Superposition only applies to linear circuits, where the output is related to the inputs by an equation of the form $y = c_1x_1 + c_2x_2 + c_3x_3 + \ldots = \sum_i c_i x_i$
- Some non-linear circuits can be considered to be approximately linear for small changes in the inputs and outputs, using the dynamic resistances of the components.
  - Dynamic resistance is the ratio of the change in voltage across a component to the change in current through the component.