4 A Short Introduction to Resistor Networks

v1.4 – June 2021

Prerequisite knowledge required: Ohm's Law and Kirchhoff's Laws

4.1 Introduction

Very often when looking at circuits, we come across combination of resistors connected together by wires: these are known as resistor networks. In many cases these resistors networks can be simplified, making analysis of the rest of the circuit quicker and easier.

In particular for this module, four such networks are worthy of particular study (and learning the equations for): resistors in series, resistors in parallel and potential and current dividers.

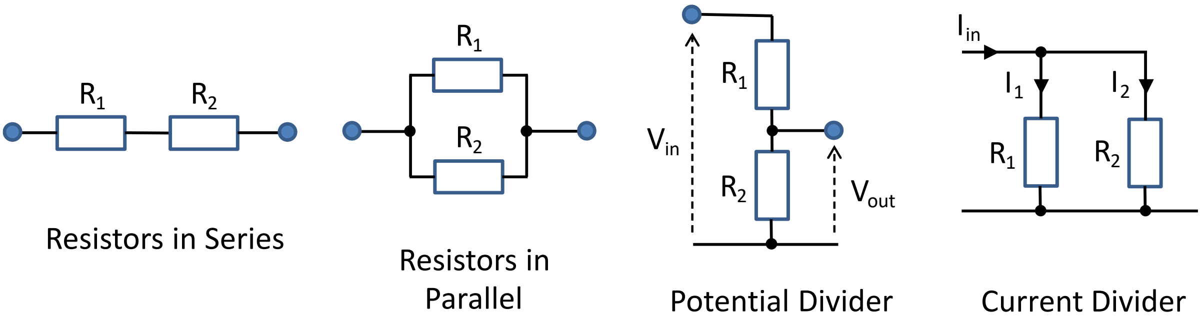

Figure 4.1 The four fundamental resistor networks

In this note I'll introduce these four networks, derive the relevant equations and give some examples of their use in the analysis of simple circuits. However as a quick reference guide, the equations you need to know are:

Resistors in series	$R = R_1 + R_2 + R_3 + \ldots$
Resistors in parallel	$R = \dfrac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3} + \ldots}$
Potential divider	$V_{out} = V_{in} \dfrac{R_2}{R_1 + R_2}$
Current divider	$I_1 = I_{in} \dfrac{R_2}{R_1 + R_2}$

4.2 Resistors in series

The simplest case of two resistors in series is shown in the diagram below, with the voltage across the two resistors R_1 and R_2 being V_1 and V_2 respectively as shown. Note that the current I is the same through both resistors[1].

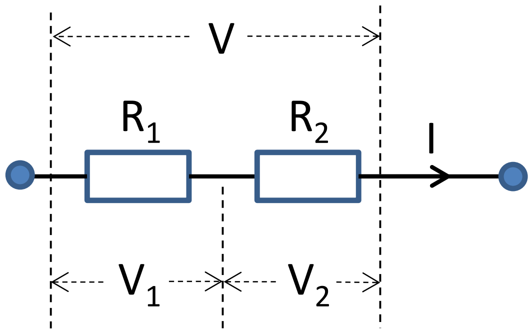

Figure 4.2 Two resistors in series

What we'd like to do is answer the question: what is the total resistance of this network? In other words, can we replace these two resistors by a single resistor that would have the same effect in a circuit, and if so, what value should that replacement resistor have?

Ohm's law states that $V = IR$ where V is the voltage across a resistor, I is the current flowing through the resistor, and R is the resistance of the resistor. Applying this law to two resistors in series (as shown) suggests that[2]:

$$\begin{aligned} V_1 &= I \times R_1 \\ V_2 &= I \times R_2 \end{aligned} \tag{4.1}$$

Just adding up these two equations gives:

$$V_1 + V_2 = I \times \left(R_1 + R_2\right) \tag{4.2}$$

Now Kirchhoff's voltage law suggests that the total voltage across both the resistors V must be the sum of the voltages across each resistor:

$$V = V_1 + V_2 \tag{4.3}$$

and substituting this into equations (4.1) gives:

$$V = I \times \left(R_1 + R_2\right) \tag{4.4}$$

[1] It has to be: just apply Kirchhoff's current law to the node in-between them.

[2] Note that the current through both resistors has to be the same by Kirchhoff's Current Law (consider the node between the two resistors: the current flowing in (through one resistor) must be equal to the current flowing out (through the other resistor)).

Now the resistance of a network with two terminals is equal to the ratio of the voltage across the network to the current flowing through it, so the resistance of this network, here called R, must be given by:

$$R = \frac{V}{I} = \frac{I \times (R_1 + R_2)}{I} = R_1 + R_2 \tag{4.5}$$

and this is a constant, independent of both the voltage and the current.

This answers the original question: yes, we can replace these two resistors by a single resistor, and the value of that resistor should be $R_1 + R_2$. This is the rule for combining two resistors in series: the total resistance is the sum of the resistance of the individual resistors.

You can extend this result to any number of resistors (although this is perhaps obvious, and a formal derivation is left as an exercise for the student).

$$R = R_1 + R_2 + R_3 + \ldots \tag{4.6}$$

4.3 Resistors in parallel

The simplest case of two resistors in parallel is shown in the diagram below, with the current through the two resistors R_1 and R_2 being I_1 and I_2 respectively as shown. The total current through both resistors is shown as I, and the voltage across both resistors is the same (as they connect the same two nodes in the circuit), and here is just called V.

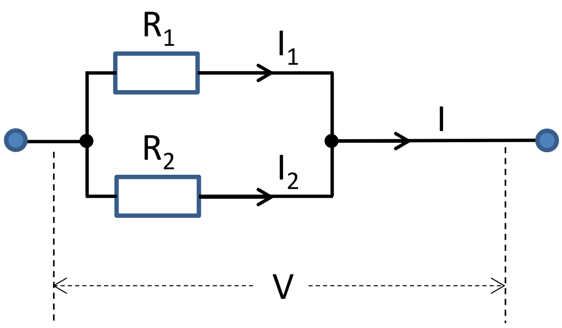

Figure 4.3 Two resistors in parallel

Again, we'd like to answer the question: what is the total resistance of this network? In other words, can we replace these two resistors by a single resistor (of value R) that has the same effect in a circuit, and if so, what value should that replacement resistor have?

This time it's the current that is split between the two resistors, not the voltage. Applying Ohm's law to each resistor now gives:

$$\begin{aligned} V &= I_1 \times R_1 \\ V &= I_2 \times R_2 \end{aligned} \tag{4.7}$$

Expressing both of these in terms of the current gives:

$$I_1 = \frac{V}{R_1} \qquad\qquad I_2 = \frac{V}{R_2} \tag{4.8}$$

and Kirchhoff's current law applied to either node in the circuit gives:

$$I = I_1 + I_2 \tag{4.9}$$

and substituting in equations (4.8) into equation (4.9) gives:

$$I = \frac{V}{R_1} + \frac{V}{R_2} = V\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{4.10}$$

Just as before the resistance of the whole network is the ratio of the voltage across the network to the total current flowing through it, which can now be easily derived, and again gives a constant, independent of the voltage and the current:

$$R = \frac{V}{I} = \frac{1}{\left(\frac{1}{R_1} + \frac{1}{R_2}\right)} \tag{4.11}$$

In general, for more than two resistors in parallel this equation can be expanded[3] as:

$$R = \frac{1}{\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} + \dots\right)} \tag{4.12}$$

and for the special case of two resistors in parallel this can also be written as:

$$R = \frac{R_1 \times R_2}{R_1 + R_2} \tag{4.13}$$

4.4 The potential divider

The next resistor network it's worth learning the formula for is the potential divider. A typical potential divider is shown in the figure below.

[3] Again, the derivation of this formula is left as an exercise for the reader. It follows the same principles as the derivation of the formula for just two resistors.

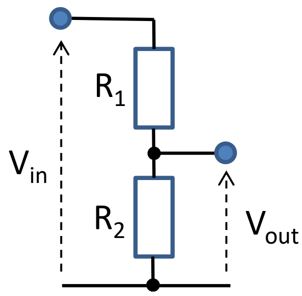

Figure 4.4 The potential divider

Here the question is: given R_1, R_2 and V_{in}, what is V_{out}?

Applying Ohm's law to the series combination of both resistors reveals that the current flowing through the two resistors is:

$$I = \frac{V_{in}}{R_1 + R_2} \tag{4.14}$$

Once the current is known, the voltage across the second resistor R_2 can be determined by just applying Ohm's law to this resistor:

$$V_{out} = I \times R_2 = V_{in} \frac{R_2}{R_1 + R_2} \tag{4.15}$$

However the output voltage is the voltage across this second resistor, so we can write that the ratio of the output to the input voltage is:

$$\frac{V_{out}}{V_{in}} = \frac{R_2}{R_1 + R_2} \tag{4.16}$$

4.5 Other combinations of resistors

Most resistor networks can be analysed in terms of the series and parallel formulas shown above, although with some circuits the process takes more than one stage. For example, consider the resistor network shown below. What is the total effective resistance of this network?

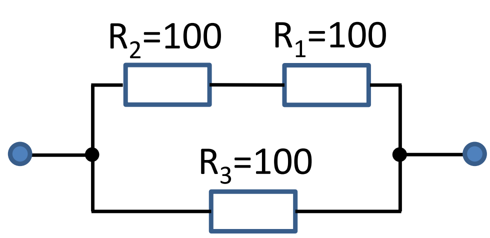

Figure 4.5 A three-resistor network

This network can be analysed in two stages. Firstly, R_1 and R_2 are a series combination of resistors, so they have an effective resistance of (R_1 + R_2). This makes this circuit the equivalent of:

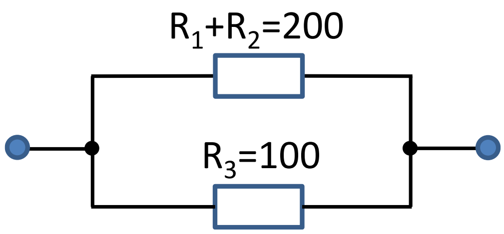

Figure 4.6 Analysing the three-resistor network

and this is just a resistance of (R_1 + R_2) in parallel with a resistor R_3, so the effective resistance of the entire network is:

$$R = \frac{1}{\frac{1}{(R_1 + R_2)} + \frac{1}{R_3}} \tag{4.17}$$

which here works out to be:

$$R = \frac{1}{\frac{1}{100+100} + \frac{1}{100}} = \frac{1}{\frac{1}{200} + \frac{1}{100}} = \frac{1}{0.005 + 0.01} = \frac{1}{0.015} = 66.7\,\Omega \tag{4.18}$$

Next, consider the potential divider circuit shown below. What is the ratio of the output voltage to the input voltage?

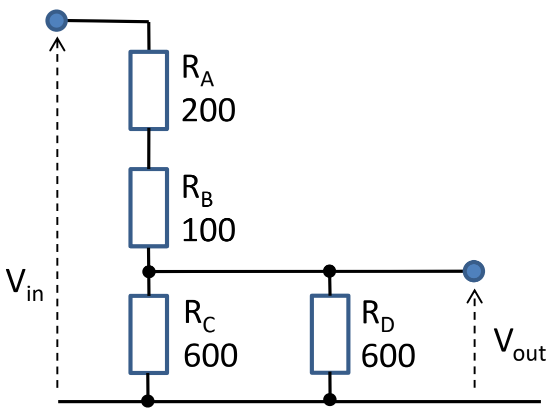

Figure 4.7 A more complex potential divider

The first stage is to combine R_A and R_B. These are resistors in series, so the total resistance of this network is:

$$R = R_A + R_B = 200 + 100 = 300\,\Omega \tag{4.19}$$

Then combine R_C and R_D. These are resistors in parallel, so their total resistance is:

$$R = \frac{1}{\frac{1}{R_C} + \frac{1}{R_D}} = \frac{1}{\frac{1}{600} + \frac{1}{600}} = \frac{1}{\left(\frac{2}{600}\right)} = \frac{1}{\left(\frac{1}{300}\right)} = 300\ \Omega \tag{4.20}$$

So in effect, this circuit is the potential divider:

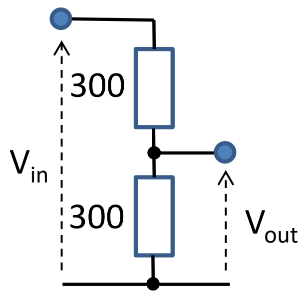

Figure 4.8 Simplifying the more complex potential divider

and applying the potential divider equation reveals that:

$$\frac{V_{out}}{V_{in}} = \frac{300}{300+300} = \frac{1}{2} \tag{4.21}$$

The output voltage is one-half of the input voltage.

4.6 A few special cases and useful facts

Perhaps the easiest special case to think about is two resistor of equal value. Suppose they are placed in parallel: what is the resistance of the resultant network?

The answer is it's half of the value of each resistor. It's a useful result to know: put two equal value resistors in series, and the result has a resistance of 2*R*; put two equal value resistors in parallel, and the result has a resistance of *R*/2.

$$\frac{1}{\frac{1}{R} + \frac{1}{R}} = \frac{1}{\left(\frac{2}{R}\right)} = \frac{R}{2} \tag{4.22}$$

Next one: two resistors in parallel always have a total resistance smaller than either of the two individual resistors. Is that statement true or false?

Answer, it's true. Consider what would have to happen if the parallel combination of R_1 and R_2 were bigger than R_1:

$$\frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} > R_1 \tag{4.23}$$

$$\frac{1}{R_1} > \frac{1}{R_1} + \frac{1}{R_2} \tag{4.24}$$

$$0 > \frac{1}{R_2} \tag{4.25}$$

which suggests that R_2 would have to be negative[4].

4.7 Dividing voltage and dividing current

Sometimes what you want to know is how much of the total voltage is dropped over each of two resistors in series, or how much of the total current goes through each of two resistors in parallel.

The first of these is easier to work out, so I'll do that one first.

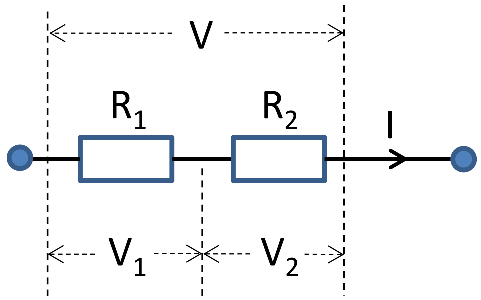

Figure 4.9 Two resistors in series (again)

The total current is the same, so we can write:

$$I = \frac{V}{R} = \frac{V_1}{R_1} = \frac{V_2}{R_2} \tag{4.26}$$

so the proportion of the voltage dropped across the first resistor R_1 is:

$$\frac{V_1}{V} = \frac{R_1}{R} = \frac{R_1}{R_1 + R_2} \tag{4.27}$$

It's equally straightforward to show that the ratio of the voltages across the two resistors is:

$$\frac{V_1}{V_2} = \frac{R_1}{R_2} \tag{4.28}$$

This is worth remembering: the ratio of the voltage across two resistors in series is equal to the ratio of the resistors.

What about currents in the case of two resistors in parallel? This is known as a current divider.

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[4] In some exceptional cases negative resistances (or at least circuits that behave as if they have negative resistance) do exist, but this is rare and we won't come across this effect in this module.

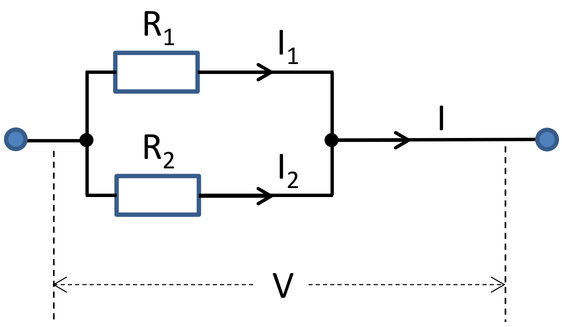

Figure 4.10 Two resistors in parallel (again)

In this case the voltage across the resistors is the same, what is interesting to work out is how much of the total current flows through each resistor. First, since the voltage across the resistors is the same, we can write:

$$V = I_1 \times R_1 = I_2 \times R_2 \tag{4.29}$$

and this quickly gives the result:

$$\frac{I_1}{I_2} = \frac{R_2}{R_1} \tag{4.30}$$

or in words: the ratio of the currents through two resistors in parallel is the inverse of the ratio of the two resistors (more current goes through the smaller resistor, less current through the bigger one).

Since the total current is the sum of I_1 and I_2 we can use Kirchhoff's current law to write $I_2 = I - I_1$ and hence derive a formula for the current through R_1 in terms of the total current and the values of the resistors:

$$\frac{I_1}{I - I_1} = \frac{R_2}{R_1} \tag{4.31}$$

$$I_1 R_1 = I R_2 - I_1 R_2 \tag{4.32}$$

$$I_1 = \frac{I \times R_2}{R_1 + R_2} \tag{4.33}$$

The current through one resistor is the value of the other resistor divided by the sum of the resistors, multiplied by the total current. This is a result worth knowing as well.

4.8 An example of using the resistor network equations

A typical digital voltmeter might have a resistance of 10 MΩ. However, placing this multimeter across a resistor to measure voltage across it won't give a precise value for the voltage across the resistor before the multimeter was added to the circuit. It is very useful to be able to work out by how much the addition to the voltmeter changes the voltages in the circuit.

Consider the circuit shown below, a simple potential divider with two 100 k resistors, and an input voltage of 3 V:

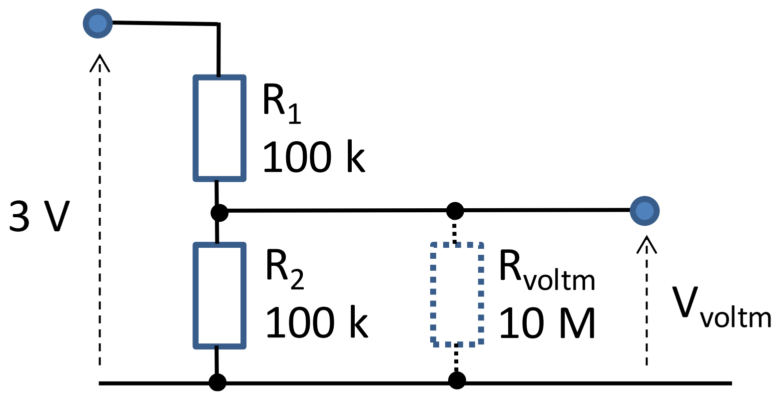

Figure 4.11 Potential divider (with a voltmeter)

Before the voltmeter is attached to the circuit (so that R_{voltm} is not there), the voltage between the two resistors R1 and R2 would clearly be 1.5 V:

$$V_{voltm} = V_{in} \times \frac{R_2}{R_1 + R_2} = 3 \times \frac{100\text{k}}{100\text{k} + 100\text{k}} = 3 \times \frac{1}{2} = 1.5\ \text{V} \tag{4.34}$$

However, any attempt to measure that 1.5 V with the voltmeter would result in the circuit having an additional 10 MΩ resistor in parallel with R_2, as shown above. Now, the output voltage as measured by the voltmeter would be:

$$V_{voltm} = V_{in} \frac{\dfrac{R_2 \times R_{voltm}}{R_2 + R_{voltm}}}{R_1 + \dfrac{R_2 \times R_{voltm}}{R_2 + R_{voltm}}} = V_{in} \frac{R_2 \times R_{voltm}}{R_1\left(R_2 + R_{voltm}\right) + R_2 \times R_{voltm}} \tag{4.35}$$

and putting the numbers into this gives:

$$\begin{aligned} V_{voltm} &= 3 \times \frac{100\text{k} \times 10\text{M}}{100\text{k}\left(100\text{k} + 10\text{M}\right) + 100\text{k} \times 10\text{M}} \\ &= 3 \times \frac{10^{12}}{10^5\left(10.1 \times 10^6\right) + 10^{12}} = 3 \times 0.4975 = 1.493\ \text{V} \end{aligned} \tag{4.36}$$

which is around 0.5% different from the voltage before the voltmeter was attached to the circuit.

This is an example of a general principle: any attempt to measure a circuit will change the operation of the circuit[5]. It's important to be able to quantify these effects so that the actual voltage across the resistor when the voltmeter is not there can be determined.

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[5] This can be particularly annoying in cases where the circuit works when the voltmeter is attached but doesn't work when the voltmeter is not there. Those sorts of circuits can be particularly difficult to debug!

4.9 Millman's theorem

Potential dividers don't always have one terminal at ground, they can look like this:

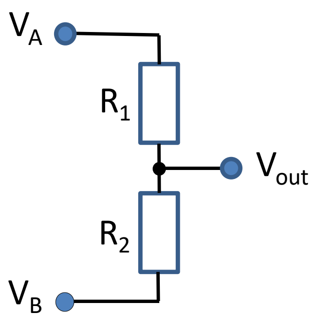

Figure 4.12 Potential divider with non-zero potentials at both ends

The analysis of this divider is quite straightforward: the current flowing through the resistors is:

$$I = \frac{(V_A - V_B)}{R_1 + R_2} \tag{4.37}$$

and therefore the voltage across the resistor R_2 is:

$$V_{out} - V_B = I \times R_2 = \frac{(V_A - V_B) \times R_2}{R_1 + R_2} \tag{4.38}$$

which makes the output voltage:

$$V_{out} = V_B + \frac{(V_A - V_B) \times R_2}{R_1 + R_2} \tag{4.39}$$

However this result can be generalise to the case where larger numbers of voltage sources and resistors all combine at a point (see Figure 4.13), and the result is known as Millman's theorem.

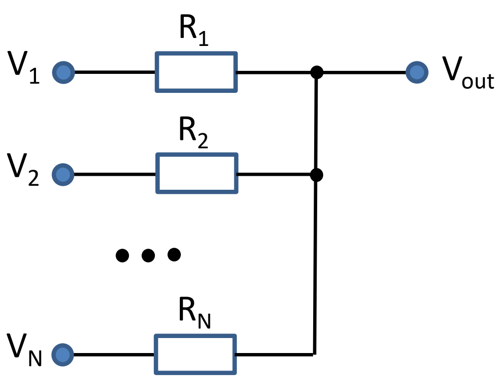

Figure 4.13 Generalised potential divider with multiple inputs

Millman's theorem states that the output voltage in this case is given by:

$$V_{out} = \frac{\sum_i \frac{V_i}{R_i}}{\sum_i \frac{1}{R_i}} \tag{4.40}$$

The easiest way to derive result this is probably to apply Kirchhoff's current law to the output node. If there is no current flowing out from the output (which is the assumption made for all the potential dividers in this chapter), then the sum of all of the currents flowing from the inputs through the resistors must be zero. Therefore:

$$\begin{aligned} \sum_i \frac{V_i - V_{out}}{R_i} &= 0 \\ \sum_i \frac{V_i}{R_i} - \sum_i \frac{V_{out}}{R_i} &= 0 \\ V_{out} \sum_i \frac{1}{R_i} &= \sum_i \frac{V_i}{R_i} \end{aligned} \tag{4.41}$$

and the result follows immediately from this. It can be useful in circuit analysis where there are a large number of resistors combining at a point.

4.10 Another useful result: the Y-Δ network transformations

Y and Δ networks are two alternative ways to represent networks of resistors with three terminals[6]. They look like this:

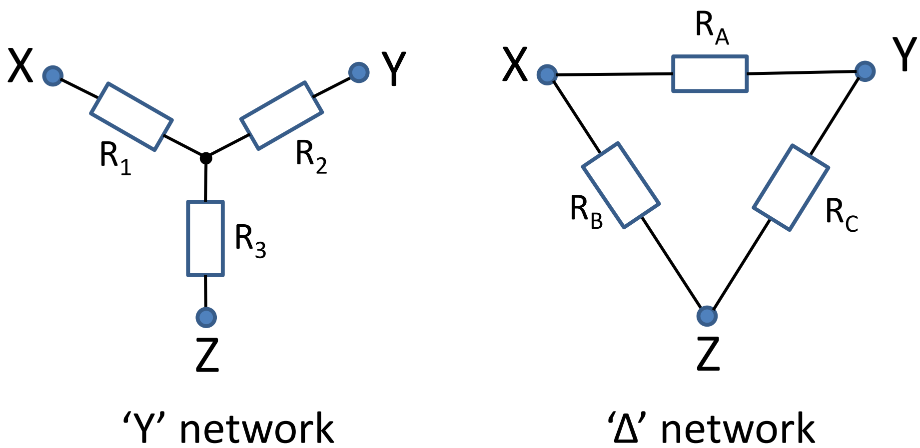

Figure 4.14 Y and Δ networks

The useful result here is that any three resistors R_1, R_2 and R_3 arranged in a Y-network behaves exactly the same as a set of three resistors R_A, R_B and R_C in a Δ-network, provided:

[6] These networks have other names as well, including T and Π networks. The idea can be generalised to more than three resistors, when the more general names "star" and "mesh" networks are often used.

$$
\begin{aligned}
R_A &= \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3} \\
R_B &= \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2} \\
R_C &= \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1}
\end{aligned} \tag{4.42}
$$

Or conversely, any three resistors arranged in a Δ-shape[7] can be transformed into an equivalent circuit with three resistors in a Y-shape, provided:

$$
\begin{aligned}
R_1 &= \frac{R_AR_B}{R_A + R_B + R_C} \\
R_2 &= \frac{R_CR_A}{R_A + R_B + R_C} \\
R_3 &= \frac{R_BR_C}{R_A + R_B + R_C}
\end{aligned} \tag{4.43}
$$

The useful results (I'll leave the derivation of these equations to the interested reader: it's not difficult, you just have to consider each of the three pairs of terminals in turn, and equate the resistances in each case).

These can be very useful results in analysing more complex networks. For example, consider the network below, with five resistors:

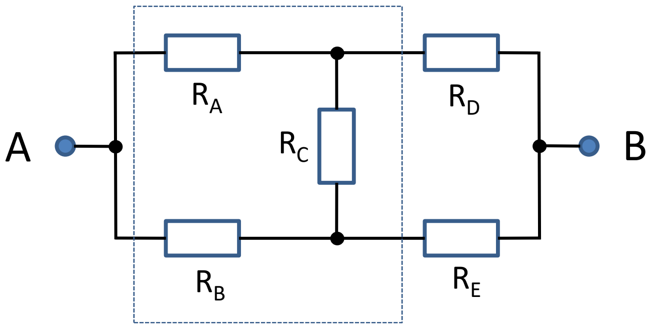

Figure 4.15 A five resistor network

It's not obvious how to analyse this on in terms of series and parallel resistors, so that the total resistance between A and B can be determined. However, take the three resistors R_A, R_B and R_C and convert them using a Δ→Y transformation, and what results is:

[7] The way I've drawn them above, you have to mentally turn the picture upside down to see why it's called a Δ network, but not why it's called a Y network. Really, the $\mathrm{Y}-\nabla$ (Y-del) or the $\lambda-\Delta$ (lambda-delta) transformation might have been a better name.

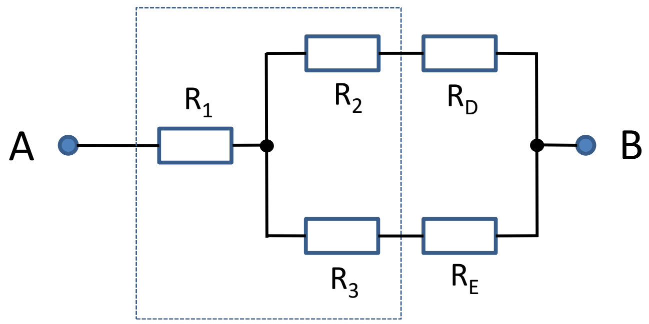

Figure 4.16 Five resistor network transformed by the Δ→Y transform

which is much easier to analyse: the total resistance R is just:

$$R = R_1 + \left(\frac{1}{R_2 + R_D} + \frac{1}{R_3 + R_E} \right)^{-1} \tag{4.44}$$

4.11 Summary: the most important things to know

- The four equations you need to know are:

Resistors in series	$R = R_1 + R_2 + R_3 + \ldots$
Resistors in parallel	$R = \dfrac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots}$
Potential divider	$V_{out} = V_{in} \dfrac{R_2}{R_1 + R_2}$
Current divider	$I_1 = I_{in} \dfrac{R_2}{R_1 + R_2}$

- Millman's theorem gives the voltage at the junction of a number of resistors given the values of the resistors and the voltages at their other ends.
    - There's no need to learn this one, it's not much harder to just apply Kirchhoff's current law and derive the result when you need it
- The Δ→Y transform can be used to convert a triangle of resistors into a star configuration. You need to know that this transform exists, where to find the formulas to work out the resistances, and when it can be useful.