3 A Short Introduction to Ohm's Law and Kirchhoff's Laws

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3.1 Introduction

Perhaps the most important skill in analogue electronics is solving circuits: from knowledge of the voltage and current sources and the components in a circuit, working out what the voltages and currents everywhere in the circuit will be.

For low-frequency circuits (the only ones we will consider here), there are three fundamental circuit laws which allow us to solve circuits: Ohm's law and Kirchhoff's voltage and current laws. These are the only equations you'll need to know to analyse a huge variety of common circuits, including most of the circuits in this module[1]. (It might save some time to learn some other equations, but all of the other equations can be derived from these three laws.)

In this note I'll introduce these laws, and explain how they can be used to analyse most simple circuits (and why they can't be used to analyse the others).

3.2 Ohm's law

Ohm's law states that the current flowing through a resistor is proportional to the voltage across it. In mathematical notation we'd write:

$$I \propto V \tag{3.1}$$

In other words, double the voltage, and the current flowing through the resistor will double as well. If we plotted a graph of the current through the resistor for different values of voltage across the terminals, it would be a straight line through the origin, something like this:

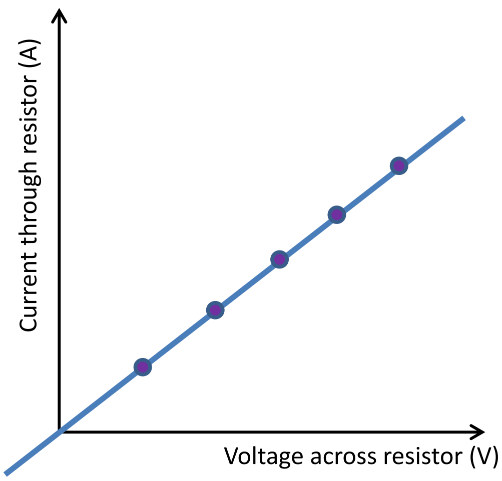

Figure 3.1 The current is proportional to the voltage across a resistor

[1] One exception is circuits with diodes in them: diodes do not obey Ohm's law and we need to use another equation which can relate the voltage across a diode to the current flowing through it.

(Note that the line extends through the origin into negative currents and negative voltages. Negative currents are perfectly possible, it just means the charge is moving the other way.)

If we use *b* to represent the gradient of this graph, we could write:

$$I = b \times V \tag{3.2}$$

Where *b* represents how easy it is for current to flow through the resistor, a quantity known as the conductance of the resistor. The higher the value of the conductance *b*, the more current flows for the same voltage. This idea of conductance didn't catch on, and it is now much more common to write this equation the other way round:

$$V = I \times \frac{1}{b} \tag{3.3}$$

and define a quantity *R* called the resistance[2] of the resistor (which is equal to 1/*b*), so that:

$$V = I \times R \tag{3.4}$$

Now a larger resistance implies a smaller conductance, and hence a smaller amount of current flowing in the circuit. The gradient of the graph above is $b = 1/R$.

Knowing two of the quantities V, I and R, you can easily work out the other one using the appropriate equation, all of which can be derived from the modern form of Ohm's law stated above in equation (3.4):

$$V = I \times R \qquad I = \frac{V}{R} \qquad R = \frac{V}{I} \tag{3.5}$$

That's about it for Ohm's law, apart from one really important (and often overlooked) point, and doing a couple of examples.

3.2.1 The really important (and often overlooked) point

The voltage across a resistor and the current flowing through the resistor are in opposite directions. Or put another way, a positive current always flows through a resistor from the higher potential to the lower potential. If you were to draw them on a circuit diagram they would look like this:

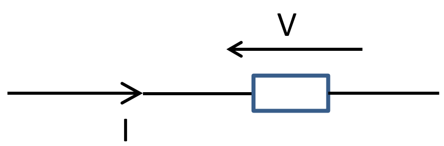

Figure 3.2 Current and voltage directions for Ohm's law

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[2] This value has units of Ohms (provided the voltage is measured in volts and the current in amps), and is named in honour of Georg Ohm who almost came up with the formula that now bears his name (see Wikipedia for the full story).

where V is the voltage across the resistor (conventionally drawn with the arrow going from the lower voltage to the higher voltage) and I is the current flowing through it. This didn't make much difference to Ohm (since all his resistors had positive values there was never any confusion about the direction of the current or voltage), but when using Ohm's law with Kirchhoff's laws in analysing circuits you do have to be careful to get the directions right. It usually helps to draw a diagram with the arrows on it.

When analysing circuits, the simplest way to make sure you get this right is to label the node in the circuit carefully, then if the voltage at node A is V_A and the voltage at node B is V_B (see Figure 3.3), Ohm's law can be written:

$$V_A - V_B = I_{AB} \times R \tag{3.6}$$

where I_{AB} is the current flowing from A to B. Notice that on both sides of the equation, A comes before B.

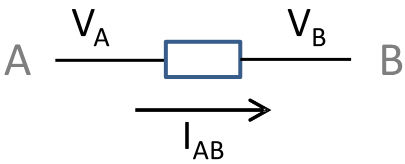

Figure 3.3 Getting the current and voltage directions right for Ohm's law

3.2.2 A couple of examples of Ohm's law

Q) A resistor has three volts across it, and a current of 1.5 mA is flowing. What is the resistance of the resistor?

A) $R = \dfrac{V}{I} = \dfrac{3}{1.5/1000} = \dfrac{3}{0.0015} = 2000\ \Omega$ (more usually written as 2kΩ)

Q) A 10k resistor has three volts across it, what current is flowing through the resistor?

A) $I = \dfrac{V}{R} = \dfrac{3}{10k} = \dfrac{3}{10 \times 1000} = \dfrac{3}{10000} = 3 \times 10^{-4} = 0.3\ \text{mA}$

3.2.3 Ohm's law and real resistors

Strictly speaking Ohm's law only applies to resistors, and resistors are components which obey Ohm's law. This is a little odd if you think about it, since it's a bit like inventing a law that says "all elephants are pink" and then saying that it only applies to pink elephants.

What Ohm found in his experiments was that for most materials he tested, his law did work[3]. And it is such a simple law, and such a good approximation for so many real components, that we use it all the time[4].

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[3] Unlike the law about pink elephants.

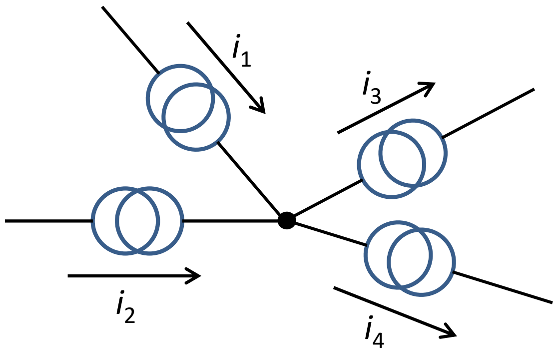
[4] See the short introduction to Electrons in Solids for one explanation of why it works so well.

However it's important to realise that there is no such thing as a perfect resistor (one that obeys Ohm's law for any applied voltage and current). For example, double the current through any real resistor and it will heat up, and as it heats up its resistance will change (often only slightly, but there is always a change). As a result of this change in resistance, the voltage across it won't exactly double.

3.3 Kirchhoff's current law

The first of Kirchhoff's two laws is probably the most obvious one. It says that the total sum of all currents flowing into any node is equal to the sum of all currents flowing out of the node. In other words, current can't just appear out of nowhere. If a current is flowing into something, an equal current must be flowing out from the other end[5].

If you think of current as a flow of charge, and know that charge cannot be created or destroyed, then this might be obvious. If you can't create or destroy charge, and wires are not elastic (in the sense that they cannot expand to accommodate more charge) then the amount of charge entering any region of the circuit must equal the amount of charge leaving that region.



$$i_1 + i_2 = i_3 + i_4$$

Figure 3.4 Illustration of Kirchhoff's current law

A bit of further thought reveals that if you can't just create a current flowing out from a point in a circuit with no current flowing into that point, then currents must flow around in circuits. This is why circuit theory is called circuit theory: currents flow around, they never just start anywhere, and they never just stop anywhere. They always flow around loops or circuits[6], and if there is no circuit, no current will be flowing.

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[5] In terms of the hydraulic analogy, this is equivalent to the statement "wires don't leak".

[6] Mathematically, we can write this as:

$$\sum_{k \in node} i_k = 0$$

which says that the sum of all currents flowing into a node is zero (here the currents that flow out from a point are considered as negative currents flowing into the node).

3.4 Kirchhoff's voltage law

The second of Kirchhoff's two laws is concerned with voltage, and states that the sum of the voltage differences across the components around any closed loop in a circuit is zero[7]. This follows from the idea of voltage being a sort of height: if you go around a circuit, sometimes going up (to a higher voltage) and sometimes going down (to a lower voltage), if you end up where you started you must have gone up and down the same amount.

When applying the law it is very important to remember which direction you are moving in, so you get the signs of the voltages right. For example, consider the circuit shown below:

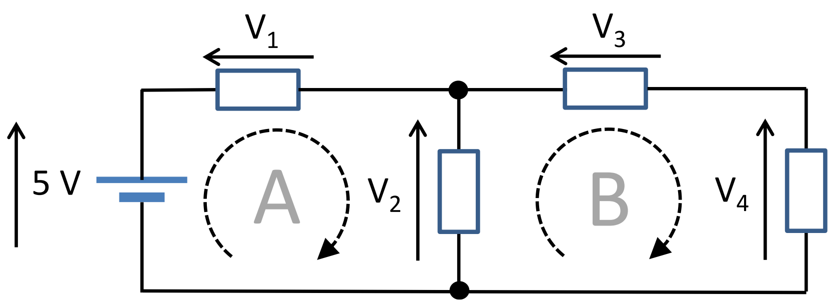

Figure 3.5 Illustration of Kirchhoff's voltage law

Around the first loop (shown as "A") the 5V from the battery is positive, since we're going clockwise and that's in the same direction as the arrow on the battery as we pass it. However the voltages across the resistors V_1 and V_2 are both negative, since they are in the opposite direction as we continue the loop through these components. Hence we can write:

$$5 - V_1 - V_2 = 0 \qquad (3.7)$$

In the second loop (shown as "B") the direction of the arrow around the loop is now in the same direction as the voltage V_2, but the opposite direction to the arrows on V_3 and V_4, and therefore:

$$V_2 - V_3 - V_4 = 0 \qquad (3.8)$$

3.4.1 Examples of Kirchhoff's laws

Q) A certain point in a circuit has four wires connected to it. A current of 1 mA flows along the first wire towards the node, and a current of 2 mA flows along the second wire towards the node. A current of 4 mA flows away from the node along the third wire. What current is flowing in the fourth wire?

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[7] Mathematically, we can write this as:

$$\sum_{k \in loop} V_k = 0$$

which just says that the sum of all the voltage differences around a loop is zero.

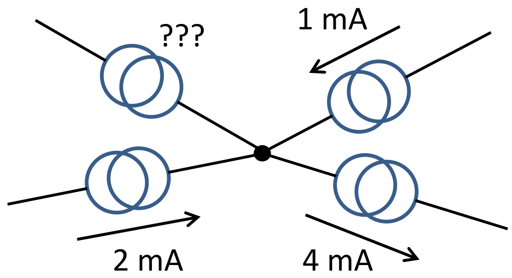

Figure 3.6 Example of Kirchhoff's current law

A) We know about 1 mA + 2 mA = 3 mA flowing into the node, and 4 mA flowing away from it. Since the total current flowing into the node must be equal to the total current flowing away, there must be another 4 mA – 3 mA = 1 mA of current flowing into the node. Hence the current in the fourth wire is 1 mA, flowing towards the node.

Q) In the figure shown below, the voltage from the battery is 5 V, and the voltage across the first resistor is 3 V. What is the voltage across the second resistor?

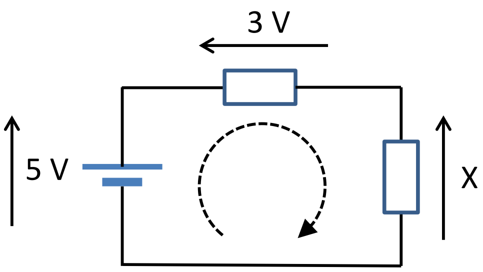

Figure 3.7 Example of Kirchhoff's voltage law

A) The voltage around the circuit must total zero, so moving around the circuit in the direction shown, the voltages are 5 V (from the battery), -3 V (negative since we're moving through the resistor in the opposite direction to the voltage shown across it, from a higher to a lower voltage) and –X V. The sum must be zero, so:

$$5+(-3)+(-X)=0$$
$$5-3-X=0$$
$$5-3=X$$
$$X=2$$

Therefore the answer is 2 V.

You could also think of this a different way: moving from the lower right corner to the upper left corner of the circuit, there are two ways you could go. If you go through the battery, you must have moved up by 5 volts. If you go through the two resistors, you must have moved up through X and then through a further 3 volts. However in both cases you've got to the same place (5 V above where you started), so 5 = X + 3, and hence X = 2.

3.4.2 Putting it all together: circuit analysis

I mentioned that Ohm and Kirchhoff's laws are all that is required to solve any circuit in this module. I'll illustrate this by a further example (many more in later notes).

Q) How much current flows through resistor R_1 in the circuit shown below?

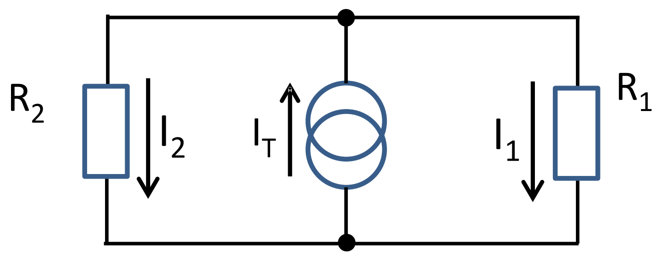

Figure 3.8 Circuit for analysis with Ohm and Kirchhoff's laws

A) There's a current source that forces a current of I_T into the node at the top of the circuit, but this current is split: some of it will flow through resistor R_1 and some through resistor R_2. How do you work out how much flows through R_1?

The key point here is that the potential difference across R_1 must be the same as the potential difference across R_2 (and indeed the potential difference across the current source), since they all join the same two nodes in the circuit.

So if we let I_1 be the current through R_1 and I_2 the current through R_2, we could use Ohm's law to write that:

$$\begin{aligned} V &= I_1 \times R_1 \\ V &= I_2 \times R_2 \\ \therefore \quad I_1 \times R_1 &= I_2 \times R_2 \end{aligned} \tag{3.9}$$

and we also know from applying Kirchhoff's current law to the top node that:

$$I_1 + I_2 = I_T \tag{3.10}$$

We've got two equations in two unknowns (I_1 and I_2). Writing $I_2 = I_T - I_1$ and substituting into equation (3.9) then gives:

$$I_1 \times R_1 = \left(I_T - I_1\right) \times R_2 \tag{3.11}$$

which after a bit of algebra reveals that:

$$I_1 = I_T \times \frac{R_2}{R_1 + R_2} \tag{3.12}$$

and similarly for the current flowing through R_2:

$$I_2 = I_T \times \frac{R_1}{R_1 + R_2} \tag{3.13}$$

Notice that this means that the current is split between the two resistors in inverse proportion to the ratio of the two resistances (in other words, $I_1 / I_2 = R_2 / R_1$).

3.5 What about circuits not in this module?

I started this note by stating that these three laws were all you needed to analyse all the circuits in this module, which does leave open the possibility that there are some circuits not covered in this module for which Ohm and Kirchhoff's laws are not enough. Is that true?

Well, yes, it is. None of these three laws work for all circuits.

3.5.1 What's wrong with Ohm's law?

The main problem with Ohm's law is that it only applies to ideal resistors. Notice the word "ideal" in that sentence. The problem is that no real resistors are ideal.

As noted above, real resistors do not obey Ohm's law exactly. While it's a very good approximation, it ignores the fact that when current passes through a resistor, energy is lost from the circuit, and dissipated as heat in the resistor. That warms up the resistor, and real resistors have temperature coefficients.

The effect is small: a typical resistor might have a temperature coefficient of around -50 ppm/°C, which means that for every degree rise in temperature, the resistance will decrease by 50 parts per million (ppm), in other words 50 / 1,000,000 * 100 = 0.005 %.

3.5.2 What's wrong with Kirchhoff's voltage law?

This one is much more interesting. Consider Faraday's experiment passing a magnet through a loop of wire: he observed a current flowing through the wire when the magnet was moved. This can be detected by an ammeter (which behaves like a small resistance).

So the circuit he used effectively looked like this:

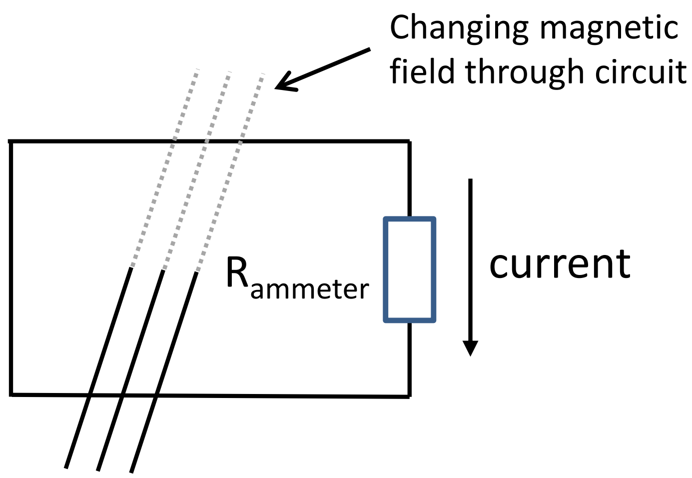

Figure 3.9 Faraday's experiment with changing magnetic fields

If you think about it, this is in direct conflict with Kirchhoff's voltage law. Current should only flow through a resistor when there is a potential difference across it, so for the ammeter to register any current, one terminal of the ammeter must be at a higher potential than the other one. But both of the terminals of the ammeter are connected together by a wire, which can have a negligible resistance, so both terminals must be at the same potential. Something is wrong.

What's going on? This turns out to be a very serious problem for Kirchhoff's voltage law, and in fact for the whole concept of voltage. The problem is that if you have any changing magnetic fields around your circuit, you cannot define the voltage at all. The whole theory of deriving a potential from the electric field breaks down.

Worse, you cannot avoid magnetic fields completely. Electromagnetic fields (light, radio waves, X-rays, etc.) are made up of a pattern of oscillating electric and magnetic fields. It is practically impossible to remove all magnetic fields from a circuit.

Fortunately, the size of the problem is a function of the size of the circuit loop, as well as the frequency and amplitude of the changing magnetic field. With small enough circuits, or at low enough frequencies, the effect is negligible.

3.5.3 What's wrong with Kirchhoff's current law?

For this one we have to refer to Einstein's theory of relativity, and in particular the result that no information can travel faster than the speed of light.

Consider a very large loop of wire, perhaps twenty kilometres all around, with a resistor, a battery and a switch at one end, and an ammeter at the other, wired up like this:

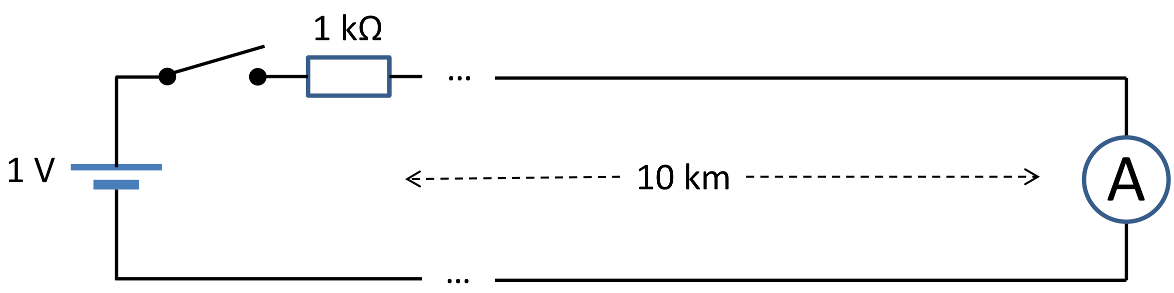

Figure 3.10 An electrically long circuit

Before the switch is closed, no current is flowing. However what happens just after the switch is closed? Kirchhoff's first law would suggest that a current of 1 V / 1k = 1 mA would immediately start to flow around the circuit, and the ammeter should register this current.

But Einstein's theory says that no information can travel faster than the speed of light. So the ammeter at the far end can't possibly register a current flowing until a time 10k / 3 x 10^8 = 33 µs after the switch is closed. What happens during that 33 µs? Einstein's theory would suggest we would have current flowing in the resistor, but not in the ammeter. Kirchhoff's current law would suggest that the current around the loop must be the same everywhere. Which is right?

In this case Einstein's theory is right. Kirchhoff's current law only applies to small circuits (or very low frequencies, so low that you're not worried about the time it takes for a signal to get from one end of the circuit to the other).

In summary: Kirchhoff's laws work well, provided the circuit is small and/or the frequency of operation of the circuit is low. This is sometimes known as the *lumped element model*, and it's valid when the period of the highest frequency of interest in the circuit is much less than the time it takes to get a signal around the circuit.

(The alternative model which deals with high-frequencies and large circuits is known as the *distributed circuit model*, but that's a subject for another module.)

3.6 Summary: the most important things to know

- There are three fundamental equations that can be used to work out the voltages and currents in any DC circuit.
- Ohm's law states that the current through a conductor is proportional to the voltage difference across it.
    - Components that (almost) obey Ohm's law are called resistors.
    - (No real component exactly obey's Ohm's law, since with more current more power is lost in the resistor, which heats it up, and resistors have slightly different values at different temperatures.)
- Kirchhoff's current law states that the total current flowing into a node in the circuit must equal the total current flowing out of the node.
    - This law is exact for DC circuits, and a very good approximation for small or low-frequency circuits.
- Kirchhoff's voltage law states that the sum of all of the voltage differences around components in a closed circuit is zero.
    - This law is not exact when there are changing magnetic fields cutting through the circuit.
    - Again, it works well for DC and small or low-frequency circuits.
- For the rest of this book, we'll assume that we are using circuits at low enough frequencies that Kirchhoff's circuit laws are sufficiently accurate for our purposes.